Non-amenability and spontaneous symmetry breaking – The hyperbolic spin-chain –

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Abstract

The hyperbolic spin chain is used to elucidate the notion of spontaneous symmetry breaking for a non-amenable internal symmetry group, here $\mathrm{SO}(1,2)$. The noncompact symmetry is shown to be spontaneously broken – something which would be forbidden for a compact group by the Mermin-Wagner theorem. Expectation functionals are defined through the $L \to \infty$ limit of a chain of length L; the functional measure is found to have its weight mostly on configurations boosted by an amount increasing at least powerlike with L. This entails that despite the non-amenability a certain subclass of non-invariant functions is averaged to an $\mathrm{SO}(1,2)$ invariant result. Outside this class symmetry breaking is generic. Performing an Osterwalder-Schrader reconstruction based on the infinite volume averages one finds that the reconstructed quantum theory is different from the original one. The reconstructed Hilbert space is nonseparable and contains a separable subspace of ground states of the reconstructed transfer operator on which $\mathrm{SO}(1,2)$ acts in a continuous, unitary, and irreducible way.

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Contents

1. Introduction

Spontaneous symmetry breaking is typically discussed for compact internal or for abelian translational symmetries, see e.g. [1, 2, 3]. Both share the property of being amenable [4]; we recall the definition below but mention already that all semisimple nonabelian noncompact Lie groups are non-amenable. The goal of this note is to elucidate the notion of spontaneous symmetry breaking for a non-amenable internal group. This is motivated by the ubiquitous appearance of noncompact internal symmetries in a gravitational context, specifically in the dimensional reduction of gravitational theories [5], further in integrable sectors of QCD [6], or in ghost- or θ-sectors of gauge theories, and also in condensed matter physics [7, 8, 9, 10, 11]. The very fact that the group is non-amenable turns out to entail a number of surprising new features. In particular spontaneous symmetry breaking becomes possible in low dimensions where it is forbidden by the Mermin-Wagner theorem [1, 12, 13] in the case of compact internal symmetries. In order to have a concrete computational framework at hand we consider a definite lattice statistical system, the hyperbolic spin chain. This is a spin chain where the dynamical variables take values in a hyperbolic (Riemannian) space of constant negative curvature and the interaction is through nearest neighbors only. The lattice formulation was chosen in order to have control over the thermodynamic limit and in preparation to the quantum field theoretical case. Indeed we expect that many of the qualitative results generalize to generic statistical systems as well as to quantum field theories. In an accompanying paper [14] we study the nonlinear sigma-model with a hyperbolic target space in 2 or more dimensions.

The systems treated always can be regarded in two different ways: either as a system of classical statistical mechanics, or as a quantum system in imaginary time. We mostly use the former interpretation, but discuss in some detail the reconstruction of the associated quantum system.

Following [3], in the quantum interpretation we consider dynamical systems $(\mathcal{C}, \tau)$ consisting of a $*$-algebra $\mathcal{C}$ ("the observables") and a one-parameter group of automorphisms ("the time evolution"), which we take to be discrete here τ^x, $x \in \mathbb{Z}$. In addition a group of automorphisms $\rho(g), g \in G$, ("the symmetry group") is supposed to act on $\mathcal{C}$ and to commute with the time evolution, $\tau \circ \rho = \rho \circ \tau$. A state ω (positive linear functional over $\mathcal{C}$) is said to be τ-invariant if $\omega \circ \tau = \omega$ and extremal τ-invariant if it is not a convex combination of different invariant states. The symmetry ρ is said to be *spontaneously broken* (see e.g. [1, 2, 3]) by an (extremal) τ-invariant state ω if $\omega \circ \rho \neq \omega$.

In the classical statistical mechanics interpretation $\mathcal{C}$ is a commutative C^*-algebra (though there may be reasons to relax this condition) and the 'time evolution' really plays the role of space translations. A symmetry is again given by a group of automorphisms $\rho(g), g \in G$, acting on $\mathcal{C}$ and leaving the Hamiltonian (or action) invariant, except for possible symmetry violating boundary condition (a very precise definition of the notion of symmetry and its spontaneous breaking can be found in [15]). The definitions of states and their invariance or noninvariance are as in the quantum interpretation. Spontaneous

symmetry breaking is then said to occur if there is an infinite volume Gibbs state (for instance obtained as a limit of finite volume Gibbs states) that is noninvariant.

We shall be interested in the above situation when the symmetry group is a non-amenable Lie group. A Lie group G is called amenable if there exists an (left) invariant state ("a mean") on the space $\mathcal{C}_b(G)$ of all continuous bounded functions on G equipped with the sup-norm. Conversely, G is called *non-amenable* if no such invariant mean over $\mathcal{C}_b(G)$ exists. All non-compact semisimple nonabelian Lie groups are known to be non-amenable. The notion of amenability has also been extended from Lie groups to homogeneous spaces (see for instance [16, 17]). Note that *if* in the above definition $\mathcal{C}$ was taken to be $\mathcal{C}_b(G)$, spontaneous symmetry breaking would be automatic for all non-amenable symmetries. We shall find however that the non-amenability also forces one to consider smaller algebras of observables (e.g. C^*-subalgebras of $\mathcal{C}_b(G)$) so that the issue becomes non-trivial again.

As a guideline it may be helpful to contrast the peculiar features we find in the hyperbolic spin chain with those in the corresponding compact model.

quantity	spherical spin-chain	hyperbolic spin chain
ground state(s)	unique, normalizable SO(3) invariant	∞ set, non-normalizable not $\mathrm{SO}(1,2)$ invariant
expectations of selected SO(2)-invariant observables	SO(3) invariant independent of bc	SO(1,2) invariant depend on bc bc selects ground state
expectations of generic non-invariant observables	SO(3) invariant independent of bc	SO(1,2) non-invariant depend on bc
reconstructed quantum theory	reproduces original one	different from original one

Here the expectations of an observable refer to the thermodynamic limit of the chain where the number of sites goes to infinity while the lattice spacing is still finite. Moreover we require that the expectations are defined through a thermodynamic limit that does not involve the selection of 'fine-tuned' subsequences. This defines a subclass of 'regular' observables to which we mostly limit the discussion. These regular observables (later called "asymptotically translation invariant") presumably include all bounded ones, but an explicit formula for their expectations can be derived regardless of boundedness. For the hyperbolic chain it turns out that one has to impose boundary conditions (bc) at the end(s) of the chain which keep at least one spin fixed. Remarkably, we find that even the expectations of invariant observables may depend on the choice of bc, even though in the limit the ends are separated by an infinite number of sites from those where the observable is supported!

The bc we are using also single out a preferred subgroup $\mathrm{SO}(2) \subset \mathrm{SO}(1,2)$ and the

expectation functionals turn out to project any observable onto its SO(2) invariant part. Since this averaging over SO(2) does not commute with the action of the full SO(1, 2) group generic non-invariant observables will signal spontaneous symmetry breaking, i.e. their expectations are not SO(1, 2) invariant. This is accompanied by an infinite family of nonnormalizable 'ground states' transforming under an irreducible representation of SO(1, 2). This representation becomes unitary under a suitable change of the scalar product; such a scalar product will be produced by the Osterwalder-Schrader reconstruction described in Section 5.

Somewhat different indications of spontaneous symmetry breaking in this context have been obtained in [18, 19]. In a situation of conventional symmetry breaking (say, of a compact Lie group symmetry in higher dimensions) one can always switch to invariant expectation functionals by performing a group average over the original noninvariant ones, at the expense of making the clustering properties worse. Here, due to the non-amenability of SO(1, 2) this cannot be done; the symmetry breaking is more severe, and in this respect resembles somewhat the 'spontaneous collapse' of supersymmetry in a spatially homogeneous state at finite temperature [20].

It is therefore remarkable that there exists a class of 'selected' SO(2) but not SO(1, 2) invariant observables (later called "SO(2) and asymptotically invariant") which get averaged to yield a SO(1, 2) invariant result. One sees that the impact of the non-amenability is quite subtle: an invariant mean for all bounded (let alone unbounded) observables cannot exist, however an invariant mean on a subalgebra does exist and can be constructed explicitly as a thermodynamic limit of probability measures. Schematically, the mechanism behind this is that for a finite chain of length L the functional measure has support mostly at configurations which are boosted with a parameter depending on and increasing with L. So provided a limit exists at all it will be SO(1, 2) invariant as all non-invariant contributions die out. This can be paraphrased by saying that the thermodynamic limit provides a *partial invariant mean*, that is a mean which is invariant only on the before-mentioned class of 'selected' noninvariant observables.

Finally we consider the counterpart of the Osterwalder-Schrader reconstruction in this context; here it is important not only to consider the regular observables but the full algebra $\mathcal{C}_b$. For the compact chain one recovers (a lattice analogue of) the quantum mechanics of a particle moving on a sphere, as expected. In the hyperbolic case, however, the reconstructed quantum theory is different from that of a particle moving on $\mathbb{H}$: whereas the former has purely continuous spectrum, the latter has at least some point spectrum. The reconstructed Hilbert space turns out to be nonseparable and the reconstructed quantum theory can be viewed as an interacting (though quantum mechanical) version of the "polymer representations" of the Weyl algebra studied in other contexts [21, 22, 23]. Consistent with these results we find that the symmetry breaking disappears in the limit of a flat target space, when the symmetry group $\mathbb{R}^2$ becomes amenable again.

For the organization of the rest of the article we refer to the table of contents.

2. The transfer matrix

The hyperbolic spin chain can be regarded as a dynamical system in the sense outlined above, with the observables being operators on a Hilbert space. On the other hand, in the classical statistical interpretation the algebra of observables is a suitable algebra of functions over (direct products of) $\mathbb{H}$ which we detail in Section 3. We represent $\mathbb{H}$ as the hyperboloid $\mathbb{H} = \{n \in \mathbb{R}^{1,2} \,|\, n \cdot n = 1\,, n^0 > 0\}$, where $a \cdot b = a^0 b^0 - a^1 b^1 - a^2 b^2$ is the bilinear form on $\mathbb{R}^{1,2}$. The time evolution of the spin chain is governed by the transfer matrix $\mathbb{T}^x$, $x \in \mathbb{N}$, which we study first. The symmetry group G is $\mathrm{SO}_0(1,2)$ which acts unitarily via the (left) quasiregular representation ρ on $L^2(\mathbb{H})$, i.e. $\rho(A)\psi(n) = \psi(A^{-1}n)$, $A \in \mathrm{SO}_0(1,2)$. Since we use the identity component exclusively we write $\mathrm{SO}(1,2)$ for $\mathrm{SO}_0(1,2)$. The time evolution commutes with the group action

$$\mathbb{T}^x \circ \rho = \rho \circ \mathbb{T}^x\,, \quad x \in \mathbb{N}\,, \tag{2.1}$$

as required. In the following we analyze the spectrum, the eigenfunctions, and the large x limit of $\mathbb{T}^x$, $x \in \mathbb{N}$, in terms of its integral kernel $\mathcal{T}_\beta(n \cdot n'; x)$. Some results from the harmonic analysis on $\mathbb{H}$ are needed which we have collected in appendix A and use freely in the following.

2.1 Spectrum and integral kernel of $\mathbb{T}^x$

The basic (1-step) transfer matrix acts on $L^2(\mathbb{H})$ and is defined by

$$(\mathbb{T}\psi)(n) = \int d\Omega(n')\, \tfrac{\beta}{2\pi} e^{\beta(1 - n\cdot n')} \psi(n')\,. \tag{2.2}$$

From (A.9), (A.20), one infers that the functions $\epsilon_{\omega,k}$ and $\epsilon_{\omega,l}$ defined in (A.8) and (A.10) are exact generalized eigenfunctions of $\mathbb{T}$ with eigenvalues

$$\lambda_\beta(\omega) = \sqrt{\frac{2\beta}{\pi}}\, e^\beta K_{i\omega}(\beta) < 1\,. \tag{2.3}$$

The eigenvalues are even functions of ω with a unique maximum at $\omega = 0$ (but only $\omega \geq$ will appear in the spectral resolution). In particular it follows that the operator $\mathbb{T}$ has absolutely continuous spectrum given by the generalized eigenvalues $\lambda_\beta(\omega)$; the spectrum covers an interval $[-q, \lambda_\beta(0)]$ with $0 < q < 1$ and is infinitely degenerate. It is interesting to note that, although real and bounded above by 1, the generalized eigenvalues are positive only for $0 < \omega < \omega_+(\beta)$, where $\omega_+(\beta)$ increases with β like $\omega_+(\beta) \sim \beta + \mathrm{const}\,\beta^{1/3}$. For $\omega > \omega_+(\beta)$ the behavior of $\lambda_\beta(\omega)$ is oscillatory with exponentially decaying amplitude

$$\lambda_\beta(\omega) \sim \sqrt{\frac{\beta}{\omega}} e^{-\frac{\pi}{2}\omega + \beta}\, 2\sin\Big[\frac{\pi}{4} + \omega\Big(\ln\frac{2\omega}{\beta} - 1\Big)\Big] \qquad \text{as} \quad \omega \to \infty\,. \tag{2.4}$$

The fact that some of the spectrum of the transfer operator is negative means that there is no reflection positivity under reflections between the lattice points. However positivity of the eigenvalues is restored in the continuum limit: introducing momentarily the lattice spacing a, physical distances $x_{\rm phys} = xa$, as well as a coupling $g^2 = 1/(\beta a)$ one has

$$\lim_{a\to 0} \,[\lambda_{\frac{1}{g^2 a}}(\omega)]^{\frac{x_{\rm phys}}{a}} = \exp\Big\{ - x_{\rm phys} \frac{g^2}{2}\Big(\frac{1}{4} + \omega^2\Big)\Big\}\,. \tag{2.5}$$

These 'eigenvalues' are readily recognized as those of the heat kernel $\exp(-\frac{g^2}{2}\mathbf{C}x_{\rm phys})$, see (A.8); $1/g$ could be removed by rescaling the n-fields; g then parameterizes the curvature of the hyperboloid.

Besides (2.4) another feature distinguishing the non-compact spin chain from the compact ones is that the iterated transfer matrix is bounded but, having continuous spectrum, is not trace class. Heuristically this is because due to the invariance (2.1) the infinite volume of $\mathrm{SO}(1,2)$ gets "overcounted" in any trace operation.

More precisely we have the following:

Lemma 2.1. *Let K be a self-adjoint operator on $L^2(\mathbb{H})$ commuting with the unitary representation ρ. Then K has only essential spectrum, implying that K cannot be compact. In particular K cannot be trace class.*

Proof. Assume that K has an eigenvalue λ. The corresponding eigenspace $\mathcal{H}_\lambda \subset L^2(\mathbb{H})$ then is invariant under the action of ρ and therefore the representation ρ can be restricted to a unitary subrepresentation ρ_λ. Since $\mathrm{SO}(1,2)$ is noncompact, ρ_λ is either infinite dimensional or it is a direct sum of copies of the trivial representation. But the trivial representation cannot be a subrepresentation of ρ since the only functions carrying the trivial representation are constants, and thus are not square integrable. ∎

Remark 1. There is a stronger version of the last statement in the proof: the trivial representation also is not even weakly contained in the direct integral decomposition of $L^2(\mathbb{H})$ because $\mathrm{SO}(1,2)$ is not amenable [4].

Remark 2. As noted above, $\mathbb{T}^x$ has only absolutely continuous spectrum.

Since $\mathbb{T}^x$ is not trace class, correlators cannot be defined by the usual expressions involving traces. The obvious remedy is gauge-fixing. This could be done by introducing a damping factor at one site and by adopting twisted boundary conditions. Then analytic computations are still feasible but are not much different from those in the simpler gauge fixing approach in which one completely freezes one spin. This is the procedure we use in section 3.

Also the iterated transfer matrix acts as an integral operator on $L^2(\mathbb{H})$ with kernel

$$\begin{aligned}
(\mathbb{T}^x\psi)(n) &= \int d\Omega(n')\mathcal{T}_\beta(n\cdot n';x)\psi(n')\,, \quad x = 1,2,3,\ldots\,,\\
\mathcal{T}_\beta(n\cdot n';x) &= \int_0^\infty \frac{d\omega}{2\pi}\,\omega\tanh\pi\omega\,\mathcal{P}_{-1/2+i\omega}(n\cdot n')\,[\lambda_\beta(\omega)]^x\,,
\end{aligned} \tag{2.6}$$

where the kernels have the semigroup property

$$\int d\Omega(n')\mathcal{T}_\beta(n\cdot n';x)\mathcal{T}_\beta(n'\cdot n'';y) = \mathcal{T}_\beta(n\cdot n'';x+y)\,. \tag{2.7}$$

Manifestly the naive expression for the trace, i.e. the $d\Omega(n)$ integral over $\mathcal{T}_\beta(1;x)$ does not exist due to the infinite volume of $\mathbb{H}$. In passing we note that in terms of the Legendre functions (2.7) amounts to the following identity ("projection property")

$$\int d\Omega(n')\,\mathcal{P}_{-1/2+i\omega}(n\cdot n')\mathcal{P}_{-1/2+i\omega''}(n'\cdot n'') = \frac{2\pi\delta(\omega-\omega'')}{\omega\tanh\pi\omega}\,\mathcal{P}_{-1/2+i\omega}(n\cdot n'')\,, \tag{2.8}$$

which can also be verified directly from (A.12). Integral kernels of spectral projections in the proper sense are easily obtained by integrating over intervals $I\ni\omega$:

$$P_I(n\cdot n') := \int_{\omega\in I}\frac{d\omega}{2\pi}\,\omega\,\tanh\pi\omega\,\mathcal{P}_{-1/2+i\omega}(n\cdot n')\,. \tag{2.9}$$

Using Eq. (2.8) one easily verifies for two intervals $I\,,J$

$$\int d\Omega(n')P_I(n\cdot n')P_J(n'\cdot n'') = P_{I\cap J}(n\cdot n'')\,, \tag{2.10}$$

showing that the operators P_I are spectral projections for an interval in ω and hence for a corresponding spectral interval for $\mathbb{T}$. Absolute continuity of the spectrum follows from the completeness relation of the generalized eigenfunctions given in appendix A.

Before proceeding let us note the continuum limit of the iterated transfer matrix. Using the notation of (2.5) one has

$$\begin{aligned}\mathcal{T}_c(\xi;g^2x_{\rm phys}) &:= \lim_{a\to0}\mathcal{T}_{\frac{1}{g^2a}}\Big(\xi;\frac{x_{\rm phys}}{a}\Big)\\ &= \int_0^\infty\frac{d\omega}{2\pi}\,\omega\,\tanh\pi\omega\,\mathcal{P}_{-1/2+i\omega}(\xi)\exp\Big\{-x_{\rm phys}\frac{g^2}{2}\Big(\frac{1}{4}+\omega^2\Big)\Big\}\,,\end{aligned} \tag{2.11}$$

where the limit is understood in the strong sense. With $t=-ix_{\rm phys}$ this is the correct result for the Feynman kernel evolving a wave function for time t, see e.g. [24, 25] and [26] for the propagators on other homogeneous spaces.. Most of the discussion on the large x limit of $\mathcal{T}_\beta(\xi;x)$ below transfers directly to the large $x_{\rm phys}$ limit of $\mathcal{T}_c(\xi,g^2x_{\rm phys})$.

Clearly for the further analysis the properties of the transfer matrix (2.6) will be crucial. By (2.2) and by iteration of the convolution property $\xi\to\mathcal{T}_\beta(\xi;x)$ is a positive function for all $x\in\mathbb{N}$ and $\beta>0$. For small x it can be evaluated explicitly

$$\begin{aligned}\mathcal{T}_\beta(\xi;0) &= \frac{1}{2\pi}\delta(\xi-1)\,,\\ \mathcal{T}_\beta(\xi;1) &= \frac{\beta}{2\pi}e^{\beta}\,e^{-\beta\xi}\,,\\ \mathcal{T}_\beta(\xi;2) &= \frac{\beta}{2\pi}e^{2\beta}\,\frac{e^{-\beta\sqrt{2(1+\xi)}}}{\sqrt{2(1+\xi)}}\,,\end{aligned} \tag{2.12}$$

with $\xi = n \cdot n' \geq 1$. The fact that $\mathcal{T}_\beta(\xi; 2)$ can be given in closed form could be used to define a coarse grained action corresponding to decimation of half of the spins. Note also the strictly monotonic decay in ξ, stronger than any power, which is masked by the rapidly oscillating integrand in (2.6). Numerical evaluation of some $x \geq 3$ transfer matrices suggests that these are generic features, see Fig. 1.

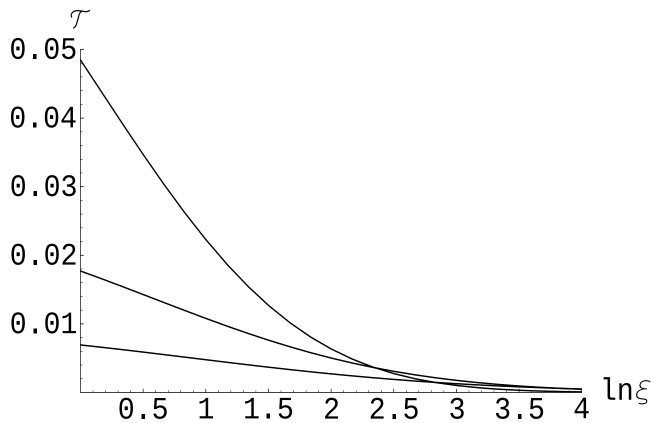

Figure 1: x-step transfer matrix $\mathcal{T}_{\beta=1}(\xi; x)$ for $x = 3, 6, 10$, in order of decreasing slope. Note the non-uniformity: $\mathcal{T}_\beta(\xi; x+1)$ is smaller/larger than $\mathcal{T}_\beta(\xi; x)$ for ξ smaller/larger than an intersection point ξ_x. By (2.26b) the enclosed area is always the same; the value at $\xi = 1$ is the x-site partition function.

We proceed to prove these and some further properties of the kernels of $\mathbb{T}^x$:

Lemma 2.2. *For fixed x the kernel $\mathcal{T}_\beta(\xi; x)$ has the following properties:*

(i) For any integer $p \geq 0$

$$\frac{1}{2\pi} t_\beta(p; x) := \int_1^\infty d\xi\, \xi^p\, \mathcal{T}_\beta(\xi; x) < \infty\,. \tag{2.13}$$

(ii) $\mathcal{T}_\beta(\xi; x)$ is strictly decreasing in ξ and vanishes for $\xi \to \infty$.

(iii) $\mathcal{T}_\beta(\xi; x) \leq \mathcal{T}_\beta(1; x)\, \mathcal{P}_{-1/2}(\xi)$ for all $\xi \geq 1$.

(iv) Let $f : [1, \infty) \to \mathbb{R}^+$ be a strictly positive locally integrable function satisfying

$$\sup_{n \cdot n^\uparrow > K} \frac{f(n \cdot n')}{f(n \cdot n^\uparrow)} \leq C (n' \cdot n^\uparrow)^p\,, \tag{2.14}$$

for some constants $p \geq 0$ and $C, K > 0$. Then

$$\left| \frac{1}{f(n \cdot n^\uparrow)} \int d\Omega(n') \mathcal{T}(n' \cdot n^\uparrow; x) f(n \cdot n') \right| \leq C'\,, \tag{2.15}$$

with some constant C'.

Remark. Condition (2.14) holds for any function f with power-like growth or decay at ∞. This follows from the fact that the geodesic distance between two points n, n' behaves aymptotically like $\ln(n \cdot n')$ and the globally valid triangle inequality for the geodesic distance on $\mathbb{H}$.

Proof. (i) The proof proceeds by induction in x. Note that naively exchanging the order of integrations in (2.6) would suggest a divergent answer already for the zero-th moment.

The point to observe is that the convolution property (2.7) implies the recursion relation

$$\begin{aligned} \mathcal{T}_\beta(\xi; x+1) &= \int_1^\infty du\, j_\beta(\xi, u)\, \mathcal{T}_\beta(u; x)\,, \\ j_\beta(\xi, u) &:= \beta e^{\beta(1-\xi\, u)} I_0(\beta\sqrt{u^2-1}\sqrt{\xi^2-1})\,, \end{aligned} \tag{2.16}$$

where $I_0(u)$ is a modified Bessel function. The kernel $j_\beta(\xi, u)$ has the following properties: its integral wrt to either variable equals 1; for fixed (not too small) ξ it is a bell-shaped function of u decaying like $\exp\{-\beta(\xi - \sqrt{\xi^2-1})u\}/\sqrt{u}$ for large u, and with a single maximum whose position grows linearly in ξ and whose value decays like $1/\xi$, for large ξ. In particular the troublesome rapidly oscillating integrand of (2.6) is gone. So in the expression defining the ξ-moments the interchange of the ξ and u integrations is legitimate. The ξ-integral can be done by repeated differentiation with respect to α of the formula ([28], p.722)

$$\int_1^\infty d\xi e^{-\alpha\xi} j_\beta(\xi, u) = \beta e^\beta \frac{e^{-\sqrt{\beta^2+2u\alpha\beta+\alpha^2}}}{\sqrt{\beta^2+2u\alpha\beta+\alpha^2}} \;=:\; F_\beta(\alpha, u)\,. \tag{2.17}$$

This will be used below to obtain explicit expressions for the low moments. Note that both sides of the equation are holomorphic functions of α for $|\alpha| < r_0 = \beta(u - \sqrt{u^2-1})$, so that we may freely differentiate at the origin. By Cauchy's estimate

$$\left|\left(-\frac{\partial}{\partial\alpha}\right)^p F_\beta(\alpha, u)\right|_{\alpha=0} \leq p!\; r^{-p} M(r, u)\;, \tag{2.18}$$

where $M(r, u)$ is the maximum of $|F|$ on the cirle $\alpha = r$, $r < r_0$. With the choice $r_1 = \beta(u - \sqrt{u^2-1/2})$ it is not hard to see that the maximum is attained for $\alpha = -r$ – this follows from the fact that the zeros of the quadratic form $Q(\alpha) := \beta + 2u\alpha\beta + \alpha^2$ are both real and negative, so both the real part and the modulus of $Q(re^{i\phi})$ take on their minimal value $\beta^2/2$ for $\phi = \pi$. One concludes from (2.18)

$$\left|\left(-\frac{\partial}{\partial\alpha}\right)^p F_\beta(\alpha, u)\right|_{\alpha=0} \leq p!\; [\beta(u - \sqrt{u^2-1/2}]^{-p}\sqrt{2}\, e^{\beta(1-1/\sqrt{2})}\;. \tag{2.19}$$

If we finally use the fact that $u - \sqrt{u^2-1/2} \geq \mathrm{const} u^{-1}$, and insert into the integral defining $t_\beta(p; x+1)$ the convolution formula (2.16) we obtain

$$t_\beta(p; x+1) \leq p!\; \mathrm{const}^p\; t_\beta(p; x)\, e^{\beta(1-1/\sqrt{2})}\,. \tag{2.20}$$

Since for $x = 1$ all moments exist trivially, this inequality shows the existence of all moments for all x and (i) is proven.

(ii) For $x = 1$ this is manifest from (2.12). For $x > 1$ we again proceed by induction. Assuming that $\mathcal{T}_\beta(\xi; x)$ is already known to be strictly decreasing, we want to show

$$\partial_\xi \mathcal{T}_\beta(\xi; x+1) = \int_1^\infty du\, \partial_\xi j_\beta(\xi, u)\, \mathcal{T}_\beta(u; x) \stackrel{!}{<} 0\,. \tag{2.21}$$

This follows from the properties of the kernel j_β, namely

$$\partial_\xi j_\beta(\xi, u) \begin{cases} < 0 & \text{for } u < u_0(\xi)\,, \\ > 0 & \text{for } u > u_0(\xi)\,, \end{cases} \qquad \text{and} \qquad \int_1^\infty du\, \partial_\xi j_\beta(\xi, u) = 0\,. \tag{2.22}$$

Using (2.22) one gets for the rhs of (2.21)

$$\begin{aligned} &\int_1^{u_0(\xi)} du\, \partial_\xi j_\beta(\xi, u) \mathcal{T}_\beta(u; x) + \int_{u_0(\xi)}^\infty du\, \partial_\xi j_\beta(\xi, u) \mathcal{T}_\beta(u; x) \\ &\qquad < \int_1^\infty du\, \partial_\xi j_\beta(\xi, u)\, \mathcal{T}_\beta(u_0(\xi); x) = 0\,, \end{aligned} \tag{2.23}$$

where in the first integral $\mathcal{T}_\beta(u; x)$ was replaced by its minimum and in the second one by its maximum over the range of integration, using the induction hypothesis. Thus $\xi \mapsto \mathcal{T}_\beta(\xi; x)$ is strictly decreasing for all x. The vanishing for $\xi \to \infty$ follows from (iii).

(iii) This is obtained from (2.6) and the estimate $|\mathcal{P}_{-1/2+i\omega}(\xi)| \leq \mathcal{P}_{-1/2}(\xi)$, which is manifest from (A.11).

(iv) The proof is an elementary consequence of (i). ■

Remark 1. Iteration of Eq. (2.16) provides an efficient way to compute numerically $\mathcal{T}_\beta(\xi; x)$ for moderately large x. This was used to produce Figs. 1 and 2.

Remark 2. Explicit expressions for the low moments are obtained by differentiating (2.17) and inserting the result in the recursion (2.16). This gives

$$\begin{aligned} p = 0: &\quad t_\beta(0; x+1) = t_\beta(0; x)\,, \\ p = 1: &\quad t_\beta(1; x+1) = \Big(1 + \frac{1}{\beta}\Big)\, t_\beta(1; x)\,, \\ p = 2: &\quad t_\beta(2; x+1) = -\frac{1}{\beta^2}(1 + \beta^2)\, t_\beta(0; x) + \frac{1}{\beta^2}(3 + 3\beta + \beta^2) t_\beta(1; x)\,, \end{aligned} \tag{2.24}$$

etc. Since for $x = 1$ all moments are known

$$t_\beta(p; 1) = \beta e^\beta (-\partial_\beta)^p (\beta e^\beta)^{-1}\,, \tag{2.25}$$

(which is basically a Laguerre polynomial in β) the x-recursions can be solved successively for $p = 0, 1, 2, \ldots$. The solution of the first two is trivial and gives

$$t_\beta(0; x) = 1\,, \qquad t_\beta(1; x) = \Big(1 + \frac{1}{\beta}\Big)^x\,, \qquad \forall\, x \in \mathbb{N}\,. \tag{2.26}$$

The higher ones won't be needed explicitly.

In summary, the qualitative properties of all the $\mathcal{T}_\beta(\xi;x)$, $x\in\mathbb{N}$, are very much like the ones exemplified in Fig. 1. The rate of decrease becomes softer with increasing x but remains faster than any power. The overall scale is set by the maximum $\mathcal{T}_\beta(1;x)$, which turns out to decrease like $x^{-3/2}$ for large x. (This is to be contrasted with the flat case of the Euclidean plane $\mathbb{R}^2$, where the decay is only like x^{-1}).

2.2 Large x asymptotics of $\mathcal{T}_\beta(\xi;x)$

We next determine the large x asymptotics of $\mathcal{T}_\beta(\xi,x)$. This is of interest because in this limit the iterated transfer matrix normally becomes a (generalized) projector onto the ground state(s), which can in particular be used to identify the latter. In a compact spin chain the kernel of the iterated transfer matrix (normalized such that the largest eigenvalue is 1) tends to a constant for $x\to\infty$, which is indeed the ground state (unique eigenstate to the highest eigenvalue) of the transfer matrix. This is a reflection of the Mermin-Wagner theorem, i.e. of the absence of spontaneous symmetry breaking. The decay in x is exponential due to the gap in the spectrum. In our noncompact model, since the spectrum is gapless, one expects the limit of large separations x to be approached power-like rather than exponentially. This is correct, but concerning the structure of the limit we are in for a surprise: the large x behavior is *not* invariant under the symmetry group $\mathrm{SO}(1,2)$. Instead one finds

$$\lim_{x\to\infty}\frac{\mathcal{T}_\beta(\xi;x)}{\mathcal{T}_\beta(1;x)}=\mathcal{P}_{-1/2}(\xi)\,, \tag{2.27}$$

as will be shown below.

So in some sense $\mathcal{P}_{-1/2}(\xi)$ plays the role of a ground state, but unlike the compact case, where there is a unique, invariant and normalizable ground state, in our case we have a whole family of generalized non-normalizable ground states $\psi_{n_0}(n)=\mathcal{P}_{-1/2}(n_0\cdot n)$, spanning a representation space of $\mathrm{SO}(1,2)$. We shall explore the consequences of (2.27) in more detail below. However already at this point it is clear that in this 1D noncompact model the Mermin-Wagner theorem cannot hold in the usual sense.

For later use we also introduce the $\mathrm{SO}^{\uparrow}(2)$ averaged versions of the iterated transfer matrix and the corresponding bounds. The former is given by

$$\overline{\mathcal{T}}_\beta(\xi,\xi';x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\varphi\,\mathcal{T}_\beta\Big(\xi\,\xi'-(\xi^2-1)^{1/2}\,(\xi'^2-1)^{1/2}\cos(\varphi-\varphi');x\Big)\,, \tag{2.28}$$

where $n=(\xi,\sqrt{\xi^2-1}\sin\varphi,\sqrt{\xi^2-1}\cos\varphi)$, etc.. Note that $\overline{\mathcal{T}}_\beta(\xi,\xi';1)=j_\beta(\xi,\xi')$ is the convolution kernel in (2.16).

We collect our results on the asymptotics of the kernels $\mathcal{T}_\beta(\xi;x)$ and $\overline{\mathcal{T}}_\beta(\xi,\xi';x)$, which contain (2.27) as a special case, in the following

Proposition 2.3. *The large x asymptotics of $\mathcal{T}_\beta(\xi;x)$ is governed by the relations:*

(i)

$$\lim_{x\to\infty} \frac{\mathcal{T}_\beta(\xi;x-y)}{\mathcal{T}_\beta(\xi';x)} = \frac{\mathcal{P}_{-1/2}(\xi)}{\mathcal{P}_{-1/2}(\xi')}\,\lambda_\beta(0)^{-y}\,. \tag{2.29}$$

(ii)

$$\left|\frac{\mathcal{T}_\beta(\xi;x)}{\mathcal{T}_\beta(1;x)} - \mathcal{P}_{-1/2}(\xi)\right| \leq \mathrm{Const}\,(\ln\xi)^2\,\mathcal{P}_{-1/2}(\xi)\,\frac{c(\beta)}{x}\,, \tag{2.30}$$

where $c(\beta)$ is given below in (2.33)

(iii)

$$\lim_{x\to\infty} \frac{\overline{\mathcal{T}}_\beta(\xi,\xi';x)}{\mathcal{T}_\beta(1;x)} = \mathcal{P}_{-1/2}(\xi)\mathcal{P}_{-1/2}(\xi')\,, \tag{2.31}$$

(iv)

$$\left|1 - \frac{\overline{\mathcal{T}}_\beta(\xi,\xi';x)}{\mathcal{T}_\beta(1;x)\mathcal{P}_{-1/2}(\xi')\mathcal{P}_{-1/2}(\xi)}\right| \leq\ [\ln^2\xi + \ln^2\xi']\,O(x^{-1})\,. \tag{2.32}$$

The main ingredient in the proof of this proposition is contained in

Lemma 2.4. *Let f be an even function of $\omega\in\mathbb{R}$, which is at least twice differentiable at 0 and grows at most polynomially as $\omega\to\infty$. Then*

$$\int_0^\infty d\omega\,\omega\,\mathrm{sh}\pi\omega\,f(\omega)\,[\tilde\lambda_\beta(\omega)]^x \sim \frac{\pi}{[c(\beta)x]^{3/2}}\left[\sqrt{\frac{\pi}{2}}f(0) + 2f'(0)[c(\beta)x]^{-1/2} + O(x^{-1})\right]$$

$$\textit{with}\quad \tilde\lambda_\beta(\omega) = \frac{\lambda_\beta(\omega)}{\lambda_\beta(0)} \quad \textit{and} \quad c(\beta) = \frac{\int_0^\infty dt\,t^2\exp(-\beta\mathrm{ch}t)}{\int_0^\infty dt\exp(-\beta\mathrm{ch}t)}\,. \tag{2.33}$$

Proof of Lemma 2.4: The idea of the proof is that the contributions of all $|\tilde\lambda_\beta(\omega)|$ for $\omega>0$ get exponentially suppressed, because they are less than 1, so only the $\omega=0$ contribution survives for $x\to\infty$. The leading power $x^{-3/2}$ arises from the double zero of the integrand at $\omega=0$ and the structure of $\tilde\lambda_\beta(\omega)$, which has a unique maximum at $\omega=0$. In more detail (2.33) one applies the Laplace expansion (see e.g. [29]) to the kernel $\exp(-xh_\beta(\omega))$, where $h_\beta(\omega) = -\ln\tilde\lambda_\beta(\omega)$ is strictly increasing in $0<\omega<\omega_+(\beta)$ with $h_\beta(0)=h'_\beta(0)=0$ and $h''_\beta(0)=c(\beta)>0$. Here $\omega_+(\beta)$ is the position of the first zero of $\lambda_\beta(\omega)$ described after Eq. (2.2). The fact that $\lambda_\beta(\omega)$ changes sign at $\omega_+(\beta)$ is inconsequential because $|\tilde\lambda_\beta(\omega)|<1$ also for $\omega\geq\omega_+(\beta)$ and the contribution of this region to the integral is exponentially suppressed. ■

Proof of Proposition 2.3: We first prove (ii). To this end set

$$\begin{aligned} D_p &:= \int_0^\infty \frac{d\omega}{2\pi}\,\omega^{1+p}\tanh\pi\omega\,[\tilde\lambda_\beta(\omega)]^x\,, \quad p=0,1,\dots\,, \\ N &:= \int_0^\infty \frac{d\omega}{2\pi}\,\omega\tanh\pi\omega\,[\tilde\lambda_\beta(\omega)]^x\,[\mathcal{P}_{-1/2+i\omega}(\xi) - \mathcal{P}_{-1/2}(\xi)]\,. \end{aligned}$$

This is chosen such that $\mathcal{T}_\beta(\xi;x)/\mathcal{T}_\beta(1;x) - \mathcal{P}_{-1/2}(\xi) = N/D_0$, as is manifest from (2.6) and $\mathcal{P}_{-1/2+i\omega}(1) = 1$. On the other hand, using the integral representation (A.11) given in Appendix A one obtains the bound

$$|\mathcal{P}_{-1/2+i\omega}(\xi) - \mathcal{P}_{-1/2}(\xi)| \leq \mathrm{Const}\, \omega^2\, \mathcal{P}_{-1/2}(\xi)\, (\ln \xi)^2\,. \tag{2.34}$$

Thus

$$\left|\frac{\mathcal{T}_\beta(\xi;x)}{\mathcal{T}_\beta(1;x)} - \mathcal{P}_{-1/2}(\xi)\right| \leq \mathrm{Const}\, (\ln \xi)^2\, \mathcal{P}_{-1/2}(\xi)\, \frac{D_2}{D_0}\,. \tag{2.35}$$

Using again Laplace's theorem, one finds $D_2/D_0 = O((c(\beta)x)^{-1})$, which establishes (ii) (and therefore also (2.27)).

(i): We apply Lemma 2.4 to $D_0 = \mathcal{T}_\beta(1;x)$ to obtain

$$\mathcal{T}_\beta(1;x) \sim \frac{\sqrt{\pi}}{[2c(\beta)\,x]^{3/2}}\, \lambda_\beta(0)^x + \ldots\,. \tag{2.36}$$

Combining (2.27) with

$$\frac{\mathcal{T}_\beta(\xi;x-y)}{\mathcal{T}_\beta(\xi';x)} = \frac{\mathcal{T}_\beta(\xi;x-y)}{\mathcal{T}_\beta(1;x-y)}\, \frac{\mathcal{T}_\beta(1;x-y)}{\mathcal{T}_\beta(1;x)}\, \frac{\mathcal{T}_\beta(1;x)}{\mathcal{T}_\beta(\xi';x)}\,, \tag{2.37}$$

and (2.36) gives (i).

(iii): This follows from averaging (2.29) over $\mathrm{SO}^\uparrow(2)$ and using (A.12c).

(iv): This is proven similarly as (2.35) starting from the spectral representation for the kernels $\overline{\mathcal{T}}_\beta$, which is obtained from (2.6) by averaging over the angles using (A.12c).

This concludes the proof of Proposition 2.3. ■

We want to mention a stronger bound for which we do not have a complete proof.

Conjecture 2.5. *The following global bound holds for all $x \in \mathbb{N}$ and for all $\xi \geq 1$:*

$$\mathcal{T}_\beta(\xi;x) \leq \mathcal{T}_\beta(1;x)\, \mathcal{P}_{-1/2}(\xi)\, E\Big(\frac{\ln \xi}{\sqrt{x}}\Big)\,, \tag{2.38}$$

for some function $E : \mathbb{R}_+ \to \mathbb{R}_+$ having finite moments of all orders.

Remark. The asymptotics of (2.35) for large x suggests that $E(t) = 1 - \frac{c_1}{c(\beta)}t^2 + O(t^3)$, with a constant c_1 of order unity and $c(\beta)$ as in (2.33). The proposal $E(t) = \exp(-\frac{c_1}{c(\beta)}t^2)$ is thus plausible. In the continuum limit (2.38) then reduces to a known global bound on the heat kernel (see e.g. [27]), noting that the geodesic distance from the origin is $\mathrm{arccosh}\xi \sim \ln \xi$ (for large ξ) and $c(\beta) \sim 1/\beta$ for large β. Given (2.38) a similar

global bound on $\overline{\mathcal{T}}_\beta(\xi,\xi';x)$ can be obtained from (2.38) by using $\xi\xi' - (\xi^2-1)^{1/2}(\xi'^2 - 1)^{1/2}\cos(\varphi-\varphi') \geq \xi[\xi' - (\xi'^2-1)^{1/2}]$, and then averaging over $\mathrm{SO}^\uparrow(2)$.

Let us now explore the consequences of (2.27) in more detail. Consider the map $\psi \mapsto P\psi$

$$(P\psi)(n) := \int d\Omega(n')\,\mathcal{P}_{-1/2}(n\cdot n')\psi(n')\,. \tag{2.39}$$

As map from $L^2(\mathbb{H})$ to itself this would have only the null vector in its domain, because it maps even strongly decreasing functions ψ into functions with a decrease so slow that they are not square integrable. But it may be regarded for instance as a map from the test function space $\mathcal{S}$ into its dual space $\mathcal{S}'$ (see appendix A). However, the range of P does not intersect its domain of definition, so the map (2.39) cannot even be iterated. This is in strong contrast to the situation in a compact model, where the corresponding operator is a well defined projection onto the 1-dimensional subspace of constant functions. Here, on the other hand, the image $(P\psi)(n)$ is in general not even invariant under some SO(2) subgroup. The Fourier transform of $P\psi$ can be defined nevertheless in a distributional sense. Using Eq. (A.14) and one finds $\omega\tanh\pi\omega\,\widehat{P\psi}(\omega,l) = 2\pi\delta(\omega)\widehat{\psi}(0,l)$, consistent with the picture that the limit (2.27) lowers the 'energy' as much as possible.

There are two further important properties that encode the ground state property of $\mathcal{P}_{-1/2}(n\cdot n')$. The first one is

Lemma 2.6. *Let K be an* SO(1, 2) *invariant integral operator with kernel $\kappa(n\cdot n')$, $\kappa\in L^1(\xi^{-1/2}\ln\xi d\xi)$. Then $P\psi$ is a generalized eigenfunction of K with eigenvalue $\widehat{\kappa}(0)$; explicitly*

$$\int d\Omega(n')\kappa(n\cdot n')\mathcal{P}_{-1/2}(n'\cdot n'') = \widehat{\kappa}(0)\mathcal{P}_{-1/2}(n\cdot n'')\,. \tag{2.40}$$

Proof. Applying the Mehler-Fock transformation (A.16) for κ and the convolution formula for the Legendre functions the left hand side becomes

$$\begin{aligned}&\int d\Omega(n')\int_0^\infty \frac{d\omega}{2\pi}\,\omega\tanh(\pi\omega)\widehat{\kappa}(\omega)\mathcal{P}_{-1/2+i\omega}(n\cdot n')\mathcal{P}_{-1/2}(n'\cdot n'')\\ &= \int_0^\infty d\omega\delta(\omega)\,\hat{\kappa}(\omega)\mathcal{P}_{-1/2+i\omega}(n\cdot n'') = \widehat{\kappa}(0)\,\mathcal{P}_{-1/2}(n\cdot n'')\,,\end{aligned} \tag{2.41}$$

(where the integral and the δ function have to be interpreted suitably to include $\omega=0$). The L^1 condition ensures that $\kappa\mathcal{P}_{-1/2}$ is integrable from 1 to ∞; this follows from the global bound $\mathcal{P}_{-1/2}(\xi)\leq(1+\ln\xi)/\sqrt{\xi}$, valid for all $\xi\geq 1$. ■

Taking for K the iterated transfer matrix one has in particular $\mathbb{T}^x P\psi = \lambda_\beta(0)^x P\psi$. For $x=1$ this gives explicitly

$$\int_1^\infty du\, j_\beta(\xi,u)\,\mathcal{P}_{-1/2}(u) = \lambda_\beta(0)\,\mathcal{P}_{-1/2}(\xi)\;, \tag{2.42}$$

using (A.21) and the fact that j_β is the $\mathrm{SO}^\uparrow(2)$ average of $\mathcal{T}_\beta(n''\cdot n;1)$. Thus $\mathcal{P}_{-1/2}$ is also an eigenfunction of the recursion relation (2.16) with the correct eigenvalue.

The second property is the 'cyclicity' of the function $\psi_\uparrow(n) := \mathcal{P}_{-1/2}(n\cdot n^\uparrow)$ for the $\mathrm{SO}^\uparrow(2)$ invariant subspace of $L^2(\mathbb{H})$ under the action of $\mathrm{SO}^\uparrow(2)$ invariant operators. See appendix A for an explicit description of this subspace and the $\mathrm{SO}^\uparrow(2)$ invariant operators acting on it. We repeat that the $\mathrm{SO}^\uparrow(2)$ denotes the stability subgroup of $n^\uparrow = (1,0,0)$. The cyclicity of $\psi_\uparrow$ then follows trivially from the fact that $\mathcal{P}_{-1/2}$ does not vanish anywhere, so any $\mathrm{SO}^\uparrow(2)$ invariant element $\psi \in L^2(\mathbb{H})$ can be obtained by acting on it with a multiplication operator. On the other hand $\psi_\uparrow$ has no nice properties with respect to operators that are not $\mathrm{SO}^\uparrow(2)$ invariant. Defining $\overline{P}$ as the integral operator with kernel $\mathcal{P}_{-1/2}(n^\uparrow\cdot n)\mathcal{P}_{-1/2}(n^\uparrow\cdot n')$ one has $(\overline{P}\psi)(n) = \mathcal{P}_{-1/2}(n^\uparrow\!\cdot n)\, C_\psi$ for some constant C_ψ. As with P one needs sufficiently strong falloff of ψ for this to be well defined and the image is again not an element of $L^2(\mathbb{H})$. As expected, $\overline{P}$ automatically projects out the part of a wave function lying in the orthogonal complement of the $\mathrm{SO}^\uparrow(2)$ invariant subspace.

3. Expectation functionals for finite length

Since the transfer operator is not trace class the overall $\mathrm{SO}(1,2)$ invariance has to be ('gauge-') fixed already for a chain of finite length. We do this by keeping the spin at one end of the chain fixed and impose various boundary conditions at the other end. Expectation functionals (mapping observables, i.e. functions of the spins into complex numbers) then are always well defined. However in the limit of infinite length interesting statements can only be made about certain subalgebras of observables which we also introduce here. As before, $\mathrm{SO}^{\uparrow}(2) \subset \mathrm{SO}(1,2)$ denotes the stability group of the vector $n^{\uparrow}$.

3.1 Boundary conditions and algebras of observables

We consider chains of length $2L+1$, with sites $x = -L, L+1, \dots L-1, L$, and spins n_x on them, in order to make the boundary go to infinity as $L \to \infty$, so as to obtain an infinite volume Gibbs state for the chosen action. As discussed earlier, some 'gauge fixing' is needed, which is accomplished conveniently by fixing the spin at the left boundary of our chain: $n_{-L} = n^{\uparrow} = (1,0,0) \in \mathbb{H}$. At the other end we consider the following choices: fixed (Dirichlet) bc $n_L = An_{-L}$, with $A \in \mathrm{SO}(1,2)$, or free bc (integrating with the invariant measure of $\mathbb{H}$ over n_L). We refer to Dirichlet bc with $A = \mathbb{1}$ as 'periodic bc' and with $A \neq \mathbb{1}$ as 'twisted bc'. The fixing of the spin n_{-L} avoids the overcounting of the infinite volume of $\mathbb{H}$ induced by the invariance (2.1); since the associated Faddeev-Popov determinant is just 1, it is justified to refer to fixed bc with $A = \mathbb{1}$ as 'periodic bc'. In some cases a nontrivial twist matrix would explicitly break the otherwise manifest $\mathrm{SO}^{\uparrow}(2)$ invariance. In those cases we shall average n_L over an $\mathrm{SO}^{\uparrow}(2)$ orbit, thereby maintaining the $\mathrm{SO}^{\uparrow}(2)$ invariance of the bc.

We consider several classes of observables, all of which consist of functions of finitely many spins. They form algebras with addition and multiplication defined pointwise. Of particular interest is the algebra $\mathcal{C}_b$ of bounded continuous functions on direct products of $\mathbb{H}$. Equipped with the sup-norm and completed with respect to this norm, this is a commutative C^*-algebra, and the expectation functionals constructed later fit the usual concept of a 'state' ω as a normalized positive (and therefore bounded) functional on the observable algebra, see e.g. [1]. More generally we consider the $*$-algebra $\mathcal{C}_p$ of polynomially bounded functions. For the construction of expectation functionals we introduce a system of subsets of $\mathcal{C}_p$, closed under a suitable norm and designed such that explicit results for thermodynamic limit can be obtained.

It turns out that the expectations of a multilocal observables $\mathcal{O} \in \mathcal{C}_p$ can always be expressed in terms of a kernel $K^{\mathcal{O}}$ associated with $\mathcal{O}$ as follows:

Definition 3.1. For $\mathcal{O} \in \mathcal{C}_p$ and $\ell \geq 2$ set

$$K^{\mathcal{O}}(n, n') := \int \prod_{i=2}^{\ell-1} d\Omega(n_i)\mathcal{O}(n_1, \dots, n_\ell) \prod_{i=2}^{\ell} \mathcal{T}_\beta(n_{i-1} \cdot n_i; x_i - x_{i-1}) \,, \tag{3.1}$$

where $n_1 = n$ and $n_\ell = n'$. For observables $\mathcal{O}$ depending only on one spin set

$$K^{\mathcal{O}}(n, n') := \mathcal{O}(n)\, \delta(n, n'), \tag{3.2}$$

where $\delta(n, n')$ is the delta-distribution (point measure) concentrated at $n = n'$, defined with respect to the measure $d\Omega$.

Lemma 3.2. *The assignment $\mathcal{O} \mapsto K^{\mathcal{O}}$ mapping observables $\mathcal{O} \in \mathcal{C}_p$ into integral operators $K^{\mathcal{O}}$ on $L^2(\mathbb{H})$ with kernel (3.1) has the following properties:*

(i) let $\mathcal{A}, \mathcal{B} \in \mathcal{C}_p$ be two observables of ordered non-overlapping 'support', i.e. $\mathcal{A}$ depends on $n_{x_1}, \dots n_{x_k}$ and $\mathcal{B}$ on $n_{x_{k+1}}, \dots n_{x_\ell}$ with $x_{k+1} \geq x_k$; then

$$\begin{aligned} &K^{\mathcal{AB}} = K^{\mathcal{A}}\, \mathbb{T}^{x_{k+1}-x_k}\, K^{\mathcal{B}} \,, \\ &K^{\mathcal{AB}}(n_1, n_\ell) = \int d\Omega(n) d\Omega(n')\, K^{\mathcal{A}}(n_1, n)\, \mathcal{T}_\beta(n \cdot n'; x_{k+1} - x_k) K^{\mathcal{B}}(n', n_\ell) \,, \end{aligned} \tag{3.3}$$

where $(\mathcal{AB})(n_{x_1}, \dots, n_{x_\ell}) = \mathcal{A}(n_{x_1}, \dots, n_{x_k})\, \mathcal{B}(n_{x_{k+1}}, \dots, n_{x_\ell})$, $k, \ell - k \geq 2$. If $x_{k+1} = x_k$, the transfer matrix $\mathcal{T}_\beta(n \cdot n'; 0)$ is interpreted as $\delta(n, n')$.

(ii) The action of $\mathrm{SO}(1,2)$ on $\mathcal{C}_p$, i.e. $\rho(A)\mathcal{O}(n_{x_1}, \dots, n_{x_l}) = \mathcal{O}(A^{-1}n_{x_1}, \dots, A^{-1}n_{x_\ell})$ induces an action on the kernels

$$K^{\rho(A)\mathcal{O}}(n_1, n_\ell) = K^{\mathcal{O}}(A^{-1}n_1, A^{-1}n_\ell) \,. \tag{3.4}$$

(iii) For the unit $1\!\!1 \in \mathcal{C}_p$ one has: $K^{1\!\!1}(n_1, n_\ell) = \mathcal{T}_\beta(n_1 \cdot n_\ell; x_\ell - x_1)$.

Proof. This is a straightforward computation.

Remark 1. The last property also implies that the correspondence $\mathcal{O} \mapsto K^{\mathcal{O}}$ is unique only for the equivalence classes obtained by inserting into a given $K^{\mathcal{O}}$ extra powers of $\mathbb{T}$. For example taking in (i) for $\mathcal{A} = 1\!\!1$ one obtains $K^{1\!\!1\mathcal{B}} = \mathbb{T}^{x_{k+1}-x_1} K^{\mathcal{B}}$. In the multipoint functions this just means that not all of the 'unobstructed' integrations have been performed. We shall therefore usually work with a reduced representative, i.e. one which cannot written in the form $\mathbb{T}^{y_1} K^{\mathcal{A}_1} \mathbb{T}^{y_2} K^{\mathcal{A}_2} \dots$ with some smaller $y_1, y_2, \dots \geq 0$.

Remark 2. For observables depending only on one spin neither (3.1) nor (3.3) are directly applicable. However the assignment $K^{\mathcal{O}}(n_1, n_2) = \mathcal{O}(n_1)\, \delta(n_1, n_2)$ is compatible with the formulas for the 1-point functions (3.13), (3.17) and the convolution (3.3), provided we associate n_1 and n_2 with the same lattice point.

We now introduce various classes of observables, where the $\mathrm{SO}^{\uparrow}(2)$ average of an observable $\mathcal{O}$ is denoted by $\overline{\mathcal{O}}$.

Definition 3.3.

(i) An observable $\mathcal{O} = \overline{\mathcal{O}} \in \mathcal{C}_p$ is called *invariant* if

$$[K^{\mathcal{O}}, \rho] = 0\,, \tag{3.5}$$

i.e. $\overline{\mathcal{O}}(An_1, \ldots, An_\ell) = \overline{\mathcal{O}}(n_1, \ldots, n_\ell)$ for all $A \in \mathrm{SO}(1,2)$.
The set of these observables is denoted by $\mathcal{C}_{\mathrm{inv}}$.

(ii) An observable $\mathcal{O} \in \mathcal{C}_p$ is called *asymptotically invariant* if

$$\lim_{A\to\infty} \rho(A)[K^{\overline{\mathcal{O}}}, \rho] = 0\ . \tag{3.6}$$

The set of these observables is denoted by $\mathcal{C}_{\mathrm{ainv}}$.

(iii) An observable $\mathcal{O} \in \mathcal{C}_p$ is called *translation invariant* if

$$[K^{\overline{\mathcal{O}}}, \mathbb{T}] = 0\ . \tag{3.7}$$

The set of these observables is denoted by $\mathcal{C}_{\mathbb{T}\,\mathrm{inv}}$.

(iv) An observable $\mathcal{O} \in \mathcal{C}_p$ is called *asymptotically translation invariant* if

$$\lim_{A\to\infty} \rho(A)[K^{\overline{\mathcal{O}}}, P] = 0\ . \tag{3.8}$$

The set of these observables is denoted by $\mathcal{C}_{\mathbb{T}\,\mathrm{ainv}}$.

In (3.8) P is the integral operator (2.39). Both in (3.6) and (3.8) $A \to \infty$ refers to a sequence of $\mathrm{SO}(1,2)$ transformations such that $\|A\| \to \infty$, and the commutator has to obey some decay condition detailed in the next section (Definitions 4.2 and 4.5).

These subsets of $\mathcal{C}_p$ are related as follows:

$$\begin{array}{ccc} \mathcal{C}_{\mathbb{T}\,\mathrm{inv}} & \subset & \mathcal{C}_{\mathbb{T}\,\mathrm{ainv}} \\ \cup & & \cup \\ \mathcal{C}_{\mathrm{inv}} & \subset & \mathcal{C}_{\mathrm{ainv}} \end{array} \tag{3.9}$$

where all inclusions are proper.

Definition 3.4. The sets $\mathcal{C}^{\uparrow}_{\mathrm{ainv}}$, $\mathcal{C}^{\uparrow}_{\mathbb{T}\,\mathrm{inv}}$ and $\mathcal{C}^{\uparrow}_{\mathbb{T}\,\mathrm{ainv}}$ are defined as the $\mathrm{SO}^{\uparrow}(2)$ invariant subsets of $\mathcal{C}_{\mathrm{ainv}}$, $\mathcal{C}_{\mathbb{T}\,\mathrm{inv}}$ and $\mathcal{C}_{\mathbb{T}\,\mathrm{ainv}}$, respectively.

Of course the inclusion relations are preserved and the counterpart of the diagram (3.9) remains valid for the $\mathrm{SO}^{\uparrow}(2)$ invariant subsets.

3.2 Expectation functionals

The expectation functionals for finite L are defined by explicitly given measures and for the largest class of observables $\mathcal{C}_p$. For states over $\mathcal{C}_b$ it follows from the general though not very constructive Banach-Alaoglu theorem [30] that thermodynamic limits always exist. The system of algebras (3.9) is designed to make useful and explicit statements about the limit, even for unbounded observables. Sometimes we refer to the expectation values as 'correlators' by a common abuse of language.

With twisted bc the finite volume average of an observable $\mathcal{O}(\{n\})$ is then defined as

$$\langle \mathcal{O} \rangle_{L,\beta,\alpha} = \frac{1}{Z_{\beta,\alpha}(2L)} \int \prod_{x=-L}^{L-1} d\Omega(n_x)\, \tfrac{\beta}{2\pi} e^{\beta(1-n_x\cdot n_{x+1})}\, \mathcal{O}(\{n\})\, \delta(n_{-L}, n^{\uparrow})\,, \tag{3.10}$$

Here we anticipate that in the cases of interest the dependence on the twist matrix A is only through the scalar product $n^{\uparrow} \cdot n_L$ or equivalently the "twist parameter" $\alpha := \mathrm{arcosh}\, n^{\uparrow} \cdot n_L \geq 0$. $Z_{\beta,\alpha}(2L)$ is the partition function normalizing the averages, $\langle 1\!\!1 \rangle_{L,\beta,\alpha} = 1$. The technique to evaluate expressions like (3.10) is well known from the compact models: one uses the semigroup property (2.7) to perform all integrations not 'obstructed' by the variables in $\mathcal{O}(\{n\})$. For the partition function there are no obstructions and one readily finds

$$Z_{\beta,\alpha}(2L) = \mathcal{T}_\beta(\mathrm{ch}\alpha; 2L)\,. \tag{3.11}$$

For the expectation value of some multilocal observable $\mathcal{O}$ one has

Proposition 3.5. (twisted bc): *For $\ell \geq 2$*

$$\begin{aligned} &\langle \mathcal{O}(n_{x_1}, \ldots n_{x_\ell}) \rangle_{L,\beta,\alpha} \\ &= \frac{1}{Z_{\beta,\alpha}(2L)} \int d\Omega(n_1) d\Omega(n_\ell)\, \mathcal{T}_\beta(n^{\uparrow} \cdot n_1; L + x_1) K^{\mathcal{O}}(n_1, n_\ell)\, \mathcal{T}_\beta(n_\ell \cdot n_L; L - x_\ell)\,, \end{aligned} \tag{3.12}$$

where $x_1 < \ldots < x_\ell$. For $\ell = 1$ we have

$$\langle \mathcal{O}(n_x) \rangle_{L,\beta,\alpha} = \frac{1}{Z_{\beta,\alpha}(2L)} \int d\Omega(n)\, \mathcal{O}(n) \mathcal{T}_\beta(n^{\uparrow} \cdot n; L + x) \mathcal{T}_\beta(n \cdot n_L; L - x)\,. \tag{3.13}$$

Proof. This is a simple consequence of (3.10) and the definition of $K^{\mathcal{O}}$. ■

Remark. As will become clear later for a $\mathrm{SO}^{\uparrow}(2)$ noninvariant field $\mathcal{O}$ one should average n_L over $\mathrm{SO}^{\uparrow}(2)$, which amounts to replacing $\mathcal{T}_\beta(n_\ell \cdot n_L; L - x_\ell)$ by $\overline{\mathcal{T}}_\beta(n^{\uparrow}\cdot n_\ell, n^{\uparrow}\cdot n_L; L - x_\ell)$ defined in (2.28). For a field $\mathcal{O}$ which is $\mathrm{SO}^{\uparrow}(2)$ invariant the replacement is an identity. Since the expectation value is taken with a positive probability measure, for observables $\mathcal{O} \in \mathcal{C}_b$ we have $|\langle \mathcal{O} \rangle| \leq ||\mathcal{O}||$ where $\|\mathcal{O}\|$ is the supremum norm, and for nonnegative $\mathcal{O}$ the expectation value is nonnegative. Observe also that due to the gauge fixing the functions (3.12) are in general not translation invariant; we shall later find a simple supplementary condition which restores translation invariance even at finite L.

For free boundary conditions at $x = L$ the situation is similar: First note that the partition function with free bc at L is

$$Z_{\beta,\text{free}}(2L) = 1. \tag{3.14}$$

This follows from the normalization $\int d\Omega(n')\, \mathcal{T}_\beta(n \cdot n'; 2L) = 1$ and the semigroup property of $\mathcal{T}_\beta(n \cdot n'; x)$, see Eqs. (2.12) and (2.7). Thus the expectation of an observable $\mathcal{O}(\{n\})$ with free bc at L is simply

$$\langle \mathcal{O} \rangle_{L,\beta,\text{free}} = \int \prod_{x=-L}^{L} d\Omega(n_x) \prod_{x=-L}^{L-1} \tfrac{\beta}{2\pi} e^{\beta(1-n_x \cdot n_{x+1})}\, \mathcal{O}(\{n\})\, \delta(n_{-L}, n^{\uparrow})\,. \tag{3.15}$$

Again these expectation values can be rewritten similarly as in Proposition 3.5:

Proposition 3.6. (free bc): *For $\ell \geq 2$*

$$\langle \mathcal{O}(n_{x_1}, \ldots n_{x_\ell}) \rangle_{L,\beta,\text{free}} = \int d\Omega(n_1) d\Omega(n_\ell)\, \mathcal{T}_\beta(n^{\uparrow} \cdot n_1; L + x_1)\, K^{\mathcal{O}}(n_1, n_\ell)\,, \tag{3.16}$$

where again $x_1 < \ldots < x_\ell$ and $K^{\mathcal{O}}$ is as in (3.1). For $\ell = 1$

$$\langle \mathcal{O}(n_x) \rangle_{L,\beta,\text{free}} = \int d\Omega(n)\, \mathcal{O}(n) \mathcal{T}_\beta(n^{\uparrow} \cdot n; L + x)\,. \tag{3.17}$$

Proof. Again a simple consequence of (3.15) and the definition of $K^{\mathcal{O}}$. ∎

Remark 1. By comparing Eqs (3.16) and (3.12) one sees that the expectation values of observables with free and twisted bc are related by

$$\int d\Omega(n_L)\, \mathcal{T}_\beta(n^{\uparrow} \cdot n_L; 2L)\, \langle \mathcal{O} \rangle_{L,\beta,\alpha} = \langle \mathcal{O} \rangle_{L,\beta,\text{free}}\,. \tag{3.18}$$

In other words for finite L the free expectation is some kind of weighted average over the twisted expectations. In the thermodynamic limit this is no longer true, as we will find below.

Remark 2. Due to (3.4) a SO$(1,2)$ transformation on the observable can always be compensated by a change in the bc

$$\langle \rho(A) \mathcal{O} \rangle_{L,\beta,\text{bc}} = \langle \mathcal{O} \rangle_{L,\beta,\text{bc}} \Big|_{n^{\uparrow} \to A^{-1} n^{\uparrow},\, n_L \to A^{-1} n_L}\,. \tag{3.19}$$

Of course our interest will be in the invariance or noninvariance of the expectations when the bc are kept fixed as $L \to \infty$.

For translation invariant observables the expectation values can be simplified. Recall that for $\mathcal{O} \in \mathcal{C}_{\mathbb{T}\,\mathrm{inv}}$

$$
\begin{aligned}
& [K^{\mathcal{O}},\, \mathbb{T}] = 0 \quad \Longleftrightarrow \quad \forall n, n' \in \mathbb{H} \\
& \int d\Omega(n')\, K^{\mathcal{O}}(n,n')\, \mathcal{T}_\beta(n' \cdot n''; 1) = \int d\Omega(n')\, \mathcal{T}_\beta(n \cdot n'; 1)\, K^{\mathcal{O}}(n', n'')\,.
\end{aligned}
\tag{3.20}
$$

For these expressions to make sense, one has to impose some technical conditions; it suffices to demand that $K^{\mathcal{O}}$ is a bounded operator. Using the convolution property (2.7) it is then easy to show that for translation invariant observables the expressions (3.12) and (3.16) simplify to

Proposition 3.7. (translation invariant observables): *For* $\mathcal{O} \in \mathcal{C}_{\mathbb{T}\,\mathrm{inv}}$

$$
\begin{aligned}
\langle \mathcal{O}(n_{x_1}, \dots n_{x_\ell}) \rangle_{L,\beta,\alpha} &= \frac{1}{Z_{\beta,\alpha}(2L)} \int d\Omega(n)\, K^{\mathcal{O}}(n^\uparrow, n) \mathcal{T}_\beta(n \cdot n_L; 2L + x_1 - x_\ell)\,, \\
\langle \mathcal{O}(n_{x_1}, \dots n_{x_\ell}) \rangle_{\beta,\mathrm{free}} &= \int d\Omega(n)\, K^{\mathcal{O}}(n^\uparrow, n)\,.
\end{aligned}
\tag{3.21}
$$

Remark 1. For twisted bc also the equivalent form of the integrand $K^{\mathcal{O}}(n, n_L)\mathcal{T}_\beta(n \cdot n^\uparrow; 2L + x_1 - x_\ell)$ could be used. Observe that these expectations are translation invariant already for finite L. Moreover for free bc they are L independent altogether, so that taking the thermodynamic limit becomes trivial.

Remark 2. For observables whose kernels admit a Fourier expansion (A.18) a necessary and sufficient condition for (3.20) to hold is that expansion takes the form

$$
K^{\mathcal{O}}(n, n') = \sum_{l,l' \in \mathbb{Z}} (-)^{l+l'} \int_0^\infty \frac{d\omega}{2\pi}\, \omega \tanh \pi\omega\, \widehat{\kappa}_{l,l'}(\omega)\, \epsilon_{\omega,-l}(n) \epsilon_{\omega,-l'}(n')\,.
\tag{3.22}
$$

It differs from the most general one in (A.18) only by the fact that it is diagonal in the energy parameter ω, as expected. An important special case is when the spectral weight is (up to a sign factor) independent of l_1, l_2. Due to the addition theorem (A.12c) the kernel becomes a function of $n_1 \cdot n_\ell$ only. In this case the corresponding observables $\overline{\mathcal{O}}$ can be characterized directly as being $\mathrm{SO}(1,2)$ invariant.

Remark 3. As already seen in section 2 the 'vacuum structure' can be explored by taking the thermodynamic limit of the discrete system. Equivalently one can first take the continuum limit and then consider its behavior for large Euclidean times. The continuum limit of the correlators in Propositions 3.5, 3.6, and 3.7 is obtained by substituting

$$
(x_i)_{\mathrm{phys}} = a x_i\,, \quad L_{\mathrm{phys}} = aL\,, \quad \beta = \frac{1}{ag^2}\,,
\tag{3.23}
$$

and taking the limit $a \to 0$. In view of (2.11) this basically amounts to replacing $\mathcal{T}_\beta$ by $\mathcal{T}_c$ everywhere, with the rescaled arguments. This procedure yields the additional bonus of restoring reflection positivity.

3.3 Projection onto $\mathrm{SO}^{\uparrow}(2)$ invariant observables

For $\mathrm{SO}(1,2)$ non-invariant observables we did not assume special symmetry properties. It turns out, however, that one needs to consider only $\mathrm{SO}^{\uparrow}(2)$ invariant observables (bounded or unbounded) since with our gauge fixing $\mathrm{SO}^{\uparrow}(2)$ noninvariant ones are effectively projected onto $\mathrm{SO}^{\uparrow}(2)$ invariant ones. In order to see this let us apply an $\mathrm{SO}^{\uparrow}(2)$ rotation $A(\varphi)$, $A(\varphi)n^{\uparrow} = n^{\uparrow}$ (with φ the rotation angle) to an $\mathrm{SO}^{\uparrow}(2)$ noninvariant observable $\mathcal{O}$. Using (3.4) one finds for $\ell \geq 2$

$$\langle \mathcal{O}(A(\varphi)n_{x_1}, \ldots, A(\varphi)n_{x_\ell})\rangle_{L,\beta,\alpha} \tag{3.24a}$$

$$= \frac{1}{Z_{\beta,\alpha}(2L)} \int d\Omega(n_1) d\Omega(n_\ell)\, \mathcal{T}_\beta(n^{\uparrow}\cdot n_1; L + x_1) K^{\mathcal{O}}(n_1, n_\ell)\, \mathcal{T}_\beta(n_\ell \cdot A(\varphi)n_L; L - x_\ell)\,,$$

$$\langle \mathcal{O}(A(\varphi)n_{x_1}, \ldots, A(\varphi)n_{x_\ell})\rangle_{L,\beta,\mathrm{free}} \tag{3.24b}$$

$$= \int d\Omega(n_1) d\Omega(n_\ell)\, \mathcal{T}_\beta(n^{\uparrow}\cdot n_1; L + x_1)\, K^{\mathcal{O}}(n_1, n_\ell)\,,$$

and similarly for $\ell = 1$. For free bc one sees that the dependence on the rotation angle drops out, so that the expectations with these bc are $\mathrm{SO}^{\uparrow}(2)$ invariant even if the observable is not. Equivalently $\mathrm{SO}^{\uparrow}(2)$ noninvariant observables have the same expectations as their $\mathrm{SO}^{\uparrow}(2)$ averages. For twisted periodic bc this is not quite true. However the $\mathrm{SO}^{\uparrow}(2)$ noninvariance of (3.24) is evidently caused by the noninvariance of the bc. To retain the $\mathrm{SO}^{\uparrow}(2)$ invariance of the bc one can average n_L over an $\mathrm{SO}^{\uparrow}(2)$ orbit. Then $\mathcal{T}_\beta$ is replaced with $\overline{\mathcal{T}}_\beta$ in Eq. (2.28) and the situation is the same as with free bc. In summary, the expectations (3.10) (when n_L is averaged over an $\mathrm{SO}^{\uparrow}(2)$ orbit) and (3.15) for finite L are already $\mathrm{SO}^{\uparrow}(2)$ invariant and hence we need not distinguish between $\mathrm{SO}^{\uparrow}(2)$ noninvariant and $\mathrm{SO}^{\uparrow}(2)$ invariant observables. In terms of the algebras introduced in section 3.1 a projection $\mathcal{C}_p \to \mathcal{C}_p^{\uparrow}$, takes place upon insertion into the expectation functionals. In terms of the kernels $K^{\mathcal{O}}$ the projection amounts to the replacement

$$K^{\mathcal{O}}(n_1, n_\ell) \;\longrightarrow\; \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi\, K^{\mathcal{O}}(A(\varphi)n_1, A(\varphi)n_\ell) =: \overline{K}^{\mathcal{O}}(n^{\uparrow}\cdot n_1, n^{\uparrow}\cdot n_\ell)\,. \tag{3.25}$$

For later reference let us issue the warning

$$\overline{K}^{\rho(A)\mathcal{O}}(n^{\uparrow}\!\cdot n_1, n^{\uparrow}\!\cdot n_\ell) \neq \overline{K}^{\mathcal{O}}(n^{\uparrow}\!\cdot A^{-1}n_1, n^{\uparrow}\!\cdot A^{-1}n_\ell)\,, \tag{3.26}$$

that is, $\mathrm{SO}^{\uparrow}(2)$ averaging does not commute with the $\mathrm{SO}(1,2)$ action.

In compact sigma models, where there is no need for gauge fixing, one can choose invariant bc, so that the expectation of any noninvariant observable is equal to that of its group average. By the Mermin-Wagner theorem in dimensions 1 and 2 this remains

also true in the thermodynamic limit, irrespective of the bc used. Here we find an analogous situation only with respect to the maximal compact subgroup $\mathrm{SO}^{\uparrow}(2)$, singled out by the gauge fixing.

In contrast, for the full $\mathrm{SO}(1,2)$ group the expectations of noninvariant and of invariant observables *cannot* be related by group averaging. This is because – due to the non-amenablity of $\mathrm{SO}(1,2)$, such averages (invariant means) do not exist [4]. Heuristically this can be understood by viewing the group averaging as a projector onto the trivial subrepresentation in the direct integral decomposition of tensor products of $L^2(\mathbb{H})$ functions. By the nonamenability the trivial representation does not occur, though. This lack of amenability is the source of many peculiarities in the vacuum structure of the noncompact model.

4. The thermodynamic limit as a partial invariant mean

By the non-amenability of SO(1,2) an invariant mean on $\mathcal{C}_b$ cannot exist; a fortiori this holds for the unbounded functions $\mathcal{C}_p$. It is known, however, that there are subspaces of the space of bounded continuous functions on any group, such as the spaces of almost periodic or weakly almost periodic functions on which a unique invariant mean exists [4]. These spaces are defined rather abstractly by relative compactness resp. weak compactness of their orbits under the group action. In the following we will introduce concretely a subspace $\mathcal{C}^{\uparrow}_{\rm ainv} \subset \mathcal{C}_p$ for which there is a unique, *invariant*, and explicitly *computable* thermodynamic limit. The infinite volume averages therefore define a 'partial invariant mean'. We presume that the bounded subalgebra $\mathcal{C}^{\uparrow}_{\rm ainv} \cap \mathcal{C}_b$ of our class $\mathcal{C}^{\uparrow}_{\rm ainv}$ (viewed as functions on SO(1,2)) consists of weakly almost periodic functions, but not of almost periodic functions (the latter set contains only the constant functions [31]). For the construction of the thermodynamic limit we proceed in several steps, where we first construct the thermodynamic limit for the algebras in the top row of the diagram (3.9). The limit is shown to be explicitly computable and unique (but different) for free and for twisted bc. The construction does not require the selection of subsequences, i.e. works without recourse to the Banach-Alaoglu theorem. In each case we then proceed to show that this limit is $\mathrm{SO}(1,2)$ invariant for the described subalgebras in the bottom row of the diagram, trivially for $\mathcal{C}_{\rm inv}$ and nontrivially for $\mathcal{C}^{\uparrow}_{\rm ainv}$.

4.1 Thermodynamic limit for translation invariant observables

We begin by studying the thermodynamic limit of translation invariant observables. The distinction between the bounded observables and the polynomially bounded observables turns out to be inessential and we assume $\mathcal{O} \in \mathcal{C}_{\mathbb{T}\,\rm inv}$ throughout. With free bc, as seen in Eq. (3.21), there is no L dependence left – so no limit has to be taken. For $\mathcal{C}_{\mathbb{T}\,\rm inv}$ expectations defined with twisted bc the existence of an $L \to \infty$ limit needs to be established.

First there is a slight complication that needs to be taken care of: twisted bc $n_L = An_{-L}$, $n_{-L} = n^{\uparrow}$, with $A \neq 1\!\!1$ explicitly break $\mathrm{SO}^{\uparrow}(2)$ invariance. Since in this study we are interested in the spontaneous symmetry breaking for the nonamenable $\mathrm{SO}(1,2)$, we restore the $\mathrm{SO}^{\uparrow}(2)$ invariance of the bc by performing an average of $n_L = An^{\uparrow}$ over $\mathrm{SO}^{\uparrow}(2)$. For finite length L the expectations will then still depend on the 'height' $n_L^0 = \mathrm{ch}\alpha$. In a slight abuse of terminology we shall keep referring to these bc as 'twisted' ones and also keep the original notation $\langle\,.\,\rangle_{L,\beta,\alpha}$. Only when a confusion is possible we emphasize the additional averaging by denoting the corresponding expectations by $\langle\,.\,\rangle_{L,\beta,\alpha,\rm av}$.

Proposition 4.1. *For $\mathcal{O} \in \mathcal{C}_{\mathbb{T}\,\rm inv}$ and twisted bc the thermodynamic limit is given by the equivalent expressions:*

$$\langle \mathcal{O}(n_{x_1}, \ldots n_{x_\ell})\rangle_{\infty,\beta,\alpha} = \lambda_\beta(0)^{x_1 - x_\ell}\, 2\pi \int_1^\infty d\xi\, \overline{K}^{O}(\xi, 1)\, \mathcal{P}_{-1/2}(\xi)\,. \tag{4.1}$$

$$\langle \mathcal{O}(n_{x_1},\ldots n_{x_\ell})\rangle_{\infty,\beta,\alpha} = \lambda_\beta(0)^{x_1-x_\ell}\, 2\pi \int_1^\infty d\xi\, \overline{K}^{\mathcal{O}}(1,\xi)\, \mathcal{P}_{-1/2}(\xi) \tag{4.2}$$

$$= \lambda_\beta(0)^{x_1-x_\ell}\, 2\pi \int_1^\infty d\xi\, \frac{\overline{K}^{\mathcal{O}}(\xi, n^\uparrow\!\cdot n_L)}{\mathcal{P}_{-1/2}(n^\uparrow\!\cdot n_L)}\, \mathcal{P}_{-1/2}(\xi)\,.$$

Proof. For Eq. (4.1) we use the $\mathrm{SO}^\uparrow(2)$ invariance of the bc to replace the transfer matrix $\mathcal{T}_\beta$ by $\overline{\mathcal{T}}_\beta$ (see (2.28)) and then $K^{\mathcal{O}}$ by its $\mathrm{SO}^\uparrow(2)$ average (see (3.25)). Since by assumption the integral $\int d\Omega(n)|K^{\mathcal{O}}(n,n')|$ exists one can in the first equation of (3.21) take the $L\to\infty$ limit inside the integral. To obtain Eq. (4.2) one uses the fact that for translation invariant observables $\mathcal{O}$ the integral operators $K^{\mathcal{O}}$ commute with P in (2.39). ■

Remark 1. There are no elements of $\mathcal{C}_{\mathbb{T}\,\mathrm{inv}}$ depending only on one spin, except constants. For free bc no thermodynamic limit has to be taken, see Proposition 3.6.

Remark 2. In particular Proposition 4.1 is valid for $\mathrm{SO}(1,2)$ invariant observables $\overline{\mathcal{O}} \in \mathcal{C}_{\mathrm{inv}} \subset \mathcal{C}_{\mathbb{T}\,\mathrm{inv}}$ where the kernel $K^{\overline{\mathcal{O}}}(n_1,n_\ell)$ depends only on the invariant distance $n_1\!\cdot n_\ell$. The thermodynamic limit (4.1) is then independent of the twist $n_{-L}\cdot n_L = n^\uparrow\cdot n_L = \cosh\alpha$, i.e.

$$\langle \overline{\mathcal{O}}(n_{x_1},\ldots n_{x_\ell})\rangle_{\infty,\beta,\alpha} = \langle \overline{\mathcal{O}}(n_{x_1},\ldots n_{x_\ell})\rangle_{\infty,\beta,0}\,, \quad \overline{\mathcal{O}} \in \mathcal{C}_{\mathrm{inv}}\,. \tag{4.3}$$

This can be verified directly using the ground state property (2.40). Indeed, if one does not take the thermodynamic limit in (3.21) with the $\mathrm{SO}^\uparrow(2)$ averaged transfer matrix one obtains initially an alternative version of the second Eq. in (4.2)

$$\langle \mathcal{O}(n_{x_1},\ldots n_{x_\ell})\rangle_{\infty,\beta,\alpha} = \frac{\lambda_\beta(0)^{x_1-x_\ell}}{\mathcal{P}_{-1/2}(\mathrm{ch}\alpha)} \int d\Omega(n)\, \overline{K}^{\overline{\mathcal{O}}}(n^\uparrow\!\cdot n)\, \mathcal{P}_{-1/2}(n\!\cdot n_L)\,, \quad \overline{\mathcal{O}} \in \mathcal{C}_{\mathrm{inv}}\,. \tag{4.4}$$

Averaging over n_L and use of the addition theorem (A.12c) shows that the dependence on α drops out. Alternatively one can use (2.40) to verify (4.3).

Remark 3. For generic translation invariant observables the infinite volume expectations are in general not $\mathrm{SO}(1,2)$ invariant. Rather one finds from (3.19) the following induced action on the kernels by $\mathcal{O} \to \rho(A^{-1})\mathcal{O}$:

$$\overline{K}^{\mathcal{O}}(1,\xi) \longrightarrow \overline{K}^{\mathcal{O}}(n^\uparrow\!\cdot An^\uparrow, n^\uparrow\!\cdot An)\,, \tag{4.5a}$$

$$\overline{K}^{\mathcal{O}}(1,\xi) \longrightarrow \frac{\mathcal{P}_{-1/2}(n^\uparrow\!\cdot An_L)}{\mathcal{P}_{-1/2}(n^\uparrow\!\cdot n_L)}\, \overline{K}^{\mathcal{O}}(n^\uparrow\!\cdot An^\uparrow, \xi) \tag{4.5b}$$

$$\overline{K}^{\mathcal{O}}(\xi, n^\uparrow\!\cdot n_L) \longrightarrow \mathcal{P}_{-1/2}(n^\uparrow\!\cdot An^\uparrow)\, \overline{K}^{\mathcal{O}}(\xi, n^\uparrow\!\cdot An_L)\,, \tag{4.5c}$$

where for free bc only (4.5a) applies while for twisted bc all three (equivalent) expressions are applicable. We shall return to these formulae later but note already here

that observables in $\mathcal{C}_{\mathbb{T}\,\mathrm{inv}} \setminus \mathcal{C}_{\mathrm{inv}}$ will in general show spontaneous symmetry breaking: $\langle \rho(A)\mathcal{O}\rangle_{\infty,\beta,\mathrm{bc}} \neq \langle \mathcal{O}\rangle_{\infty,\beta,\mathrm{bc}}$.

For the rest of this subsection we now focus on the special case of SO(1, 2) invariant observables. Then symmetry breaking is not an issue, nevertheless the result (4.1) is surprising. Besides the mere existence of a thermodynamic limit one would of course expect that the effect of the different bc is washed out. While we have found that the dependence on the twist chα actually does disappear, free bc in general give a different thermodynamic limit. In other words, even *invariant* observables show a *dependence on the boundary conditions*, even after the boundary is removed to infinity!

To illustrate this consider specifically the usual 'spin-spin' two-point functions with the various bc. For twisted bc the thermodynamic limit is obtained from (4.1) and (3.1) (for $\ell = 2$ with $\overline{\mathcal{O}}(n_1, n_2) = n_1 \cdot n_2$) as

$$\lim_{L\to\infty} \langle n_0 \cdot n_x \rangle_{L,\beta,\alpha} = \lim_{L\to\infty} \langle n_0 \cdot n_x \rangle_{L,\beta,0} = \frac{2\pi}{\lambda_\beta(0)^x} \int_1^\infty d\xi\, \xi\, \mathcal{T}_\beta(\xi; x) \mathcal{P}_{-1/2}(\xi)\,. \tag{4.6}$$

The independence of the twist angle has been seen before to be a general feature. However the same expectation with free bc at the right end of the chain gives a different result. One finds

$$\langle n_0 \cdot n_x \rangle_{\beta,\mathrm{free}} = 2\pi \int_1^\infty d\xi\, \xi\, \mathcal{T}_\beta(\xi; x) = \Big(1 + \frac{1}{\beta}\Big)^x, \tag{4.7}$$

using Eq. (2.26b) in the second step. As seen generally in Eq. (3.21) the correlator is L-independent and thus coincides with its thermodynamic limit. But this thermodynamic limit is now different from the previous one. To make sure that the analytical expressions (4.6) and (4.7) really define different functions we evaluated them numerically; the results are shown in Figure 2 below. For periodic bc also the approach to the thermodynamic limit is shown, which turns out to be nonuniform and extremely slow.

For the 'internal energy' $E_{\beta,\mathrm{bc}} := \lim_{L\to\infty} \langle n_0 \cdot n_1 \rangle_{L,\beta,\mathrm{bc}}$ the discrepancy can be seen immediately:

$$\begin{aligned} E_{\beta,\mathrm{bc}} &= 1 + \frac{1}{\beta} - \lim_{L\to\infty} \frac{1}{2L} \frac{\partial}{\partial\beta} \ln Z_{\beta,\mathrm{bc}}(2L) \\ &= \begin{cases} 1 + \dfrac{1}{\beta} - \dfrac{\partial}{\partial\beta} \ln \lambda_\beta(0) & \text{for twisted bc,} \\ 1 + \dfrac{1}{\beta} & \text{for free bc,} \end{cases} \end{aligned} \tag{4.8}$$

using Eq. (2.36) and (3.14), respectively.

Technically the discrepancy can be traced back to the fact that in Eq. (3.18) the operations 'averaging' and 'taking the thermodynamic limit' do not commute, schematically: $\lim_{L\to\infty} \int d\Omega(n_L)(\ldots) \neq \int d\Omega(n_L) \lim_{L\to\infty}(\ldots)$. Indeed, the lhs is L-independent and equals $\langle \overline{\mathcal{O}} \rangle_{\beta,\mathrm{free}}$ while the integrand and hence the integral on the rhs vanishes pointwise.

In fact the integrand on the right hand side behaves very nonuniformly for $L \to \infty$: for instance the two-point function with twisted bc is unbounded as a function of α and the convergence as $L \to \infty$ takes place more and more slowly as α increases.

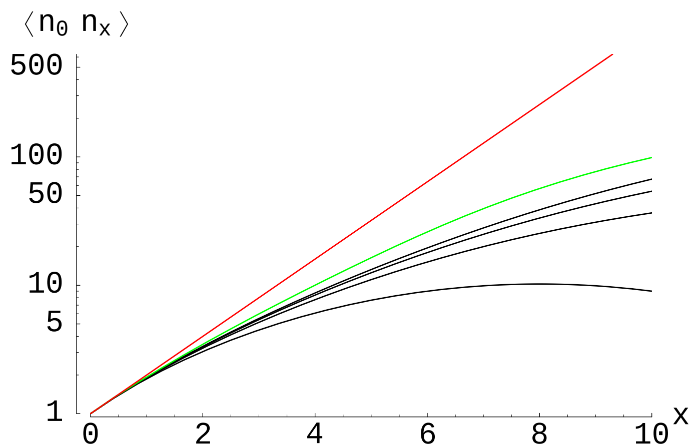

Figure 2: Spin two-point function for $\beta = 1$: for periodic bc and $L = 8, 16, 32, 64, \infty$, and for free bc, in order of increasing values at fixed x.

These features are in sharp contrast to those of the compact O(N) spin chains where it is well known that all boundary conditions yield the *same* thermodynamic limit for the correlators of invariant as well as noninvariant quantities; see for instance [32]. In the compact models no gauge fixing is required, but one could fix a spin at the boundary just as we did here, and the thermodynamic limit would be insensitive to it. This is a consequence of the Mermin-Wagner theorem, which holds in this case.

One might suspect that this 'long range order' in the non-compact model reflects the poor choice of observables, i.e. that the kernel $\overline{\mathcal{O}}(n, n') = n \cdot n'$ does not define an operator on $L^2(\mathbb{H})$ (as explained after Eq. (A.13)). However the situation is the same for invariant kernels $\overline{\mathcal{O}}(n, n') = \kappa(n \cdot n')$ which obey (A.17) and which therefore do define integral operators on $L^2(\mathbb{H})$. The thermodynamic limit of the corresponding two-point functions is obtained simply by replacing $\xi \mathcal{T}_\beta(\xi; x)$ with $\kappa(\xi)\mathcal{T}_\beta(\xi; x)$ in Eqs. (4.6) and (4.7). These two-point functions will be conventional, decreasing functions of x. Nevertheless they will in general be different for free and for periodic bc.

Another potential problem could be the lack of clustering. However for SO(1, 2) invariant observables the situation turns out to be peculiar – there is perfect clustering even at finite distance. Consider two invariant observables, $\mathcal{A}(n_{x_1}, \dots n_{x_\ell})$ and $\mathcal{B}(n_{y_1}, \dots n_{y_k})$ such that $x_1 < \dots < x_\ell \le y_1 < \dots < y_k$. We claim that for all bc

$$\langle \mathcal{A}\mathcal{B} \rangle_{\infty,\beta,\mathrm{bc}} = \langle \mathcal{A} \rangle_{\infty,\beta,\mathrm{bc}} \langle \mathcal{B} \rangle_{\infty,\beta,\mathrm{bc}} \,. \tag{4.9}$$

For twisted periodic bc the derivation proceeds along the lines leading to Eq. (4.3) via (4.4): we define kernels $K^{\mathcal{A}}$ and $K^{\mathcal{B}}$ as above and use the ground state property

Eq. (2.40) of $\mathcal{P}_{-1/2}$. It turns out that both sides of Eq. (4.9) are equal to the same multiple of $\widehat{\kappa}_{\mathcal{A}}(0)\widehat{\kappa}_{\mathcal{B}}(0)$ (and thus in particular are independent of the twist parameter). For free bc the expectations of invariant observables are already L-independent; the asserted factorization can be seen in a way similar to the step from (3.16) to (3.21).

This 'hyperclustering' property is unpleasant, because it means that from the correlators of invariant fields one can only reconstruct a one-dimensional Hilbert space. The latter is suggested by the fact that all vectors obtained by applying invariant kernels to the ground state will by (2.40) be proportional to it. Technically it follows from the Osterwalder-Schrader reconstruction of the Hilbert space, as detailed in section 5. On the other hand this feature is a peculiarity present likewise for other one-dimensional spin models, like the compact O(N) chains or the harmonic chains. In these models, since they are based on amenable symmetries, there exists a unique thermodynamic limit also for noninvariant correlators and therefore one obtains by the reconstruction a nontrivial infinite dimensional Hilbert space. We now show that in the noncompact models the situation encountered for invariant observables persists for a class of noninvariant ones: for all observables in $\mathcal{C}_{\mathbb{T}\,\mathrm{ainv}}$ for fixed bc a unique thermodynamic limit exists but is in general different for periodic and for free bc. The hyperclustering, however, does not carry over to those observables, as we will see.

4.2 TD limit for asymptotically translation invariant observables

We now relax the condition of translation invariance to "asymptotic translation invariance". It suffices to consider $\mathrm{SO}^{\uparrow}(2)$ invariant bc (such as free, periodic, or $\mathrm{SO}^{\uparrow}(2)$ averaged twisted bc). As explained in section 3.3 this allows one to restrict attention to $\mathrm{SO}^{\uparrow}(2)$ invariant observables. As before we denote by $K^{\mathcal{O}}$ and P the integral operators with kernels $K^{\mathcal{O}}(n,n')$ in (3.1) and $\mathcal{P}_{-1/2}(n\cdot n')$, respectively. Similarly $[K^{\mathcal{O}},P](n,n')$ is the kernel of the commutator of $K^{\mathcal{O}}$ with P.

We give now the precise version of Definition 3.3 (iv):

Definition 4.2. $\mathcal{O}\in\mathcal{C}_p$ is called *asymptotically translation invariant* iff its $\mathrm{SO}^{\uparrow}(2)$ average satisfies

$$\Big|[K^{\overline{\mathcal{O}}},P](n,n')\Big| \leq p(n^{\uparrow}\cdot n)\,p(n^{\uparrow}\cdot n')\,,\qquad p(\xi)\sim \xi^{-1/2}(\ln\xi)^{-3}\,,\tag{4.10}$$

for some fixed n and all $n'\in\mathbb{H}$ or vice versa. For observables $\mathcal{O}(n)$ depending on a single spin only we define asymptotic translation invariance by the condition that their $\mathrm{SO}^{\uparrow}(2)$ average $\overline{\mathcal{O}}(n)$ has a limit as $n\to\infty$.

The function $p(\xi)$ needs to be bounded but it is mainly the large ξ asymptotics that matters; for definiteness we take $p(\xi)=p_1\,\xi^{-1/2}(1+\ln\xi)^{-3}$, for some $p_1=p(1)>0$.

To motivate the terminology "asymptotically translation invariant" recall from section 2 that P can be viewed as a weak limit of transfer operators $\mathbb{T}^L/\mathcal{T}_\beta(1;L)$ for $L\to\infty$.

Further, for $\mathcal{O} \in \mathcal{C}_{\mathbb{T}\,\mathrm{ainv}}$ one has

$$\lim_{A\to\infty} \rho(A)[K^{\mathcal{O}}, P] = 0\,. \tag{4.11}$$

Here we assumed that the commutator acts on L^1 wave functions so that $\rho(A)([K^{\mathcal{O}}, \mathbb{T}]\psi)(n)$ can be bounded by $\frac{\beta}{2\pi} p_1 p(A n^\uparrow \cdot n)\|\psi\|_1$.

The thermodynamic limit for asymptotically translation invariant multi-spin observables and twisted bc is given by the *same* expressions as for translation invariant observables:

Proposition 4.3.

(i) Let $\mathcal{O} \in \mathcal{C}_{\mathbb{T}\,\mathrm{ainv}}$ be a 1-point observable, i.e. any function of one spin such that its $\mathrm{SO}^\uparrow(2)$ *average has a limit $\overline{\mathcal{O}}(\infty)$. Then:*

$$\lim_{L\to\infty} \langle \mathcal{O}(n_x)\rangle_{L,\beta,\mathrm{bc}} = \overline{\mathcal{O}}(\infty)\,. \tag{4.12}$$

(ii) Let $\mathcal{O} \in \mathcal{C}_{\mathbb{T}\,\mathrm{ainv}}$ be a multi-point observable, $\ell \geq 2$. If Conjecture 2.5 holds then:

$$\langle \mathcal{O}(n_{x_1}, \ldots, n_{x_\ell})\rangle_{\infty,\beta,\alpha} = \lambda_\beta(0)^{x_1 - x_\ell}\, 2\pi \int_1^\infty d\xi\, \overline{K}^{O}(\xi, 1)\, \mathcal{P}_{-1/2}(\xi)\,. \tag{4.13}$$

Proof. (i) By definition the $\mathrm{SO}^\uparrow(2)$ average of $\mathcal{O}(n)$ has a limit $\overline{\mathcal{O}}(\infty)$ for $n \to \infty$ (and therefore is a bounded function). We write

$$\langle \mathcal{O}(n_x)\rangle_{L,\beta,\mathrm{bc}} = \overline{\mathcal{O}}(\infty) + \int d\mu_{L,\beta,\mathrm{bc}}(n; x)[\mathcal{O}(n) - \overline{\mathcal{O}}(\infty)]\,. \tag{4.14}$$

Decomposing the second term into an integral over $n^\uparrow \cdot n \in [1, \Lambda]$ and $n^\uparrow \cdot n \in [\Lambda, \infty[$, given ϵ choose Λ so large that $\sup|\overline{\mathcal{O}}(\overline{\infty}) - \mathcal{O}(n)| < \epsilon$, with the supremum over $n^\uparrow \cdot n \in [\Lambda, \infty[$. In Lemma 4.7 below is shown that the 1-spin measure of any bounded set in $\mathbb{H}$ goes to 0 as $L \to \infty$, so sending $L \to \infty$ the first integral vanishes. This shows that the total integral goes to 0 for $L \to \infty$ and one obtains (4.12).

(ii) The proof is based on a reduction to the case (i) of a one-spin observable. It is convenient to write $(AB)(n, n')$ for the kernel of AB, for any pair of integral operators A, B. With this notation one starts from

$$\langle \mathcal{O}\rangle_{L,\beta,\alpha} = \int d\Omega(n_\ell)\, (\mathbb{T}^{L+x_1} K^{\mathcal{O}})(n^\uparrow, n_\ell)\, \frac{\overline{\mathcal{T}}_\beta(n^\uparrow \cdot n_\ell, n^\uparrow \cdot n_L; L - x_\ell)}{\mathcal{T}_\beta(n^\uparrow \cdot n_L; 2L)}\,. \tag{4.15}$$

These multipoint averages can be written as one-point averages as follows

$$\langle \mathcal{O}\rangle_{L,\beta,\alpha} = \langle \mathcal{O}_{0,L,\alpha}\rangle_{L,\beta,\alpha}\,, \quad \text{with}$$

$$\mathcal{O}_{0,L,\alpha}(n_0) := \int d\Omega(n)\, (\mathbb{T}^{x_1} K^{\mathcal{O}})(n_0, n)\, \frac{\overline{\mathcal{T}}_\beta(n^\uparrow \cdot n, n^\uparrow \cdot n_L; L - x_\ell)}{\overline{\mathcal{T}}_\beta(n^\uparrow \cdot n_0, n^\uparrow \cdot n_L; L)}\,. \tag{4.16}$$

For the time being (4.16) is just an identity (Fubini's theorem); later we shall put it in the context of the Osterwalder-Schrader reconstruction. Next we observe that $\mathcal{O}_{0,L,\alpha}(n_0)$ has a $L \to \infty$ limit, pointwise for all $n_0 \in \mathbb{H}$, which is independent of the twist parameter α defining n_L modulo $\mathrm{SO}^{\uparrow}(2)$ rotations:

$$\lim_{L\to\infty} \mathcal{O}_{0,L,\alpha}(n_0) = \mathcal{O}_{0,\infty}(n_0)\,, \quad \text{with}$$
$$\mathcal{O}_{0,\infty}(n_0) := \int d\Omega(n)\,(\mathbb{T}^{x_1}K^{\mathcal{O}})(n_0,n)\frac{\mathcal{P}_{-1/2}(n^{\uparrow}\cdot n)}{\mathcal{P}_{-1/2}(n^{\uparrow}\cdot n_0)}\,\lambda_\beta(0)^{-x_\ell}\,. \tag{4.17}$$

Here we used Eqs. (2.31). The crucial identity now is

Lemma 4.4. *Assume that Conjecture 2.5 holds. Then*

$$\lim_{L\to\infty}\langle \mathcal{O}_{0,L,\alpha}\rangle_{L,\beta,\alpha} = \lim_{L\to\infty}\langle \mathcal{O}_{0,\infty}\rangle_{L,\beta,\alpha}\,, \quad \textit{for all}\;\; \mathcal{O}\in\mathcal{C}_b\,. \tag{4.18}$$

Proof of Lemma 4.4: We start with the bound

$$\begin{aligned}&\Big|\langle(\mathcal{O}_{0,L,\alpha}-\mathcal{O}_{0,\infty})\rangle_{L,\beta,\alpha}\Big| \qquad (4.19)\\ &\leq \int d\Omega(n_0)\,\Big|\mathcal{O}_{0,L,\alpha}(n_0)-\mathcal{O}_{0,\infty}(n_0)\Big|\,\frac{\mathcal{T}_\beta(n_0\cdot n^{\uparrow};L)\overline{\mathcal{T}}_\beta(n^{\uparrow}\!\cdot n_0, n^{\uparrow}\!\cdot n_L;L)}{\mathcal{T}_\beta(n^{\uparrow}\!\cdot n_L;2L)}\,,\end{aligned}$$

To examine the difference $|\mathcal{O}_{0,L,\alpha}(n_0)-\mathcal{O}_{0,\infty}(n_0)|$ we write

$$\mathcal{O}_{0,L,\alpha}(n_0)-\mathcal{O}_{0,\infty}(n_0) = \mathcal{D}_1(n_0)+\mathcal{D}_2(n_0) \tag{4.20}$$

with

$$\begin{aligned}\mathcal{D}_1(n_0) &:= \mathcal{O}_{0,L,\alpha}(n_0)-\mathcal{E}(n_0)\\ \mathcal{D}_2(n_0) &:= \mathcal{E}(n_0)-\mathcal{O}_{0,\infty}(n_0)\end{aligned} \tag{4.21}$$

and

$$\mathcal{E}(n_0) := \int d\Omega(n)(\mathbb{T}^{x_1}K^{\mathcal{O}})(n_0,n)\,\frac{\overline{\mathcal{T}}_\beta(n^{\uparrow}\!\cdot n;n^{\uparrow}\!\cdot n_L;L-x_\ell)}{\mathcal{P}_{-1/2}(n^{\uparrow}\cdot n_0)\mathcal{P}_{-1/2}(n^{\uparrow}\!\cdot n_L)\mathcal{T}_\beta(1;L)}. \tag{4.22}$$

Using the bound

$$\Big|(\mathbb{T}^{x_1}K^{\mathcal{O}})(n_0,n)\Big| \leq \|\mathcal{O}\|\,\mathcal{T}(n_0\cdot n;x_\ell) \tag{4.23}$$

and the convolution property of the transfer matrices, a bound for $\mathcal{D}_1$ is

$$|\mathcal{D}_1(n_0)| \leq \|\mathcal{O}\|\left|1-\frac{\overline{\mathcal{T}}_\beta(n^{\uparrow}\!\cdot n_0,n^{\uparrow}\!\cdot n_L;L)}{\mathcal{P}_{-1/2}(n^{\uparrow}\cdot n_0)\mathcal{P}_{-1/2}(n^{\uparrow}\!\cdot n_L)\mathcal{T}_\beta(1;L)}\right|, \tag{4.24}$$

while for $\mathcal{D}_2$ one obtains simply

$$
\begin{aligned}
|\mathcal{D}_2(n_0)| &\leq \|\mathcal{O}\| \int d\Omega(n) \, \frac{\mathcal{T}_\beta(n_0 \cdot n; x_\ell)}{\mathcal{P}_{-1/2}(n^\uparrow \cdot n_0)} \\
&\qquad \times \left| \mathcal{P}_{-1/2}(n^\uparrow \cdot n)\lambda_\beta(0)^{-x_\ell} - \frac{\overline{\mathcal{T}}_\beta(n^\uparrow \!\cdot n, n^\uparrow \!\cdot n_L; L - x_\ell)}{\mathcal{P}_{-1/2}(n^\uparrow \!\cdot n_L)\mathcal{T}_\beta(1; L)} \right| . \qquad (4.25)
\end{aligned}
$$

According to (4.19) we have to estimate

$$
d_{1,2} := \int d\Omega(n_0) \, |\mathcal{D}_{1,2}(n_0)| \, \frac{\mathcal{T}_\beta(n_0 \cdot n^\uparrow; L)\overline{\mathcal{T}}_\beta(n^\uparrow \!\cdot n_0, n^\uparrow \!\cdot n_L; L)}{\mathcal{T}_\beta(n^\uparrow \!\cdot n_L; 2L)} \, , \qquad (4.26)
$$

In a first step we use $\overline{\mathcal{T}}_\beta(\xi_0, \xi_L; x) \leq \mathcal{T}_\beta(1; x)\mathcal{P}_{-1/2}(\xi_0)\mathcal{P}_{-1/2}(\xi_L)$ and the fact that $\mathcal{D}_{1,2}(n_0)$ depend on $\xi_0 = n^\uparrow \!\cdot n_0$ only to write

$$
d_{1,2} \leq 2\pi \frac{\mathcal{P}_{-1/2}(\xi_L)\mathcal{T}_\beta(1; L)}{\mathcal{T}_\beta(\xi_L; 2L)} \int_1^\infty d\xi_0 |\mathcal{D}_{1,2}(\xi_0)| \mathcal{T}_\beta(\xi_0, L)\mathcal{P}_{-1/2}(\xi_0) \, . \qquad (4.27)
$$

Next we claim

$$
|\mathcal{D}_{1,2}(\xi_0)| \leq [\ln^2 \xi_0 + \ln^2 \xi_L] O(1/L) \, . \qquad (4.28)
$$

For $\mathcal{D}_1(\xi_0)$ this follows directly from (4.24) and Proposition 2.3(iv). For $\mathcal{D}_2(\xi_0)$ we likewise use Proposition 2.3(iv) and then apply Lemma 2.2(iv) with the function $f(\xi) = \mathcal{P}_{-1/2}(\xi)[\ln^2 \xi + \ln^2 \xi_L]$ (see the remark after Lemma 2.2). This gives

$$
\begin{aligned}
|\mathcal{D}_2(n_0)| &\leq O(1/L) \int_1^\infty d\xi \, \overline{\mathcal{T}}_\beta(\xi_0, \xi; x_\ell) \frac{\mathcal{P}_{-1/2}(\xi)}{\mathcal{P}_{-1/2}(\xi_0)} [\ln^2 \xi + \ln^2 \xi_L] \\
&\leq O(1/L)[\ln^2 \xi_0 + \ln^2 \xi_L] \, , \qquad (4.29)
\end{aligned}
$$

as asserted.

On account of (4.28) the integrand $I(\xi_0)$ in (4.27) vanishes pointwise for $L \to \infty$. Using (4.28), assuming Conjecture 2.5 and recalling that $\mathcal{P}_{-1/2}(\xi_0) \leq (1 + \ln \xi_0)/\sqrt{\xi_0}$ we can bound $I(\xi_0)$ by $O(1/L)\mathcal{T}_\beta(1; L)\xi_0^{-1}(1 + \ln \xi_0)^2 E(\ln \xi_0/\sqrt{L})[\ln^2 \xi_0 + \ln^2 \xi_L]$. Changing now the integration variable to $t := (\ln \xi_0)/\sqrt{L}$ the new integrand is bounded by

$$
F(t) := \text{const } (t + \frac{1}{\sqrt{L}})^2 \; (t^2 + \frac{\ln^2 \xi_L}{L}) E(t) \qquad (4.30)
$$

and the right hand side is bounded uniformly in L by an integrable function. By the dominated convergence theorem we can interchange the limit with the integration and conclude that $d_{1,2} \to 0$ for $L \to \infty$, completing the proof of Lemma 4.4. ■

This lemma, combined with (4.16), reduces the computation of the thermodynamic limit for multipoint functions to that of one-point functions: from Eqs. (4.12), (4.16) it follows that the thermodynamic limit of a multipoint observable can be computed as

$$\lim_{L\to\infty} \langle \mathcal{O} \rangle_{L,\beta,\alpha} = \lim_{n_0\to\infty} \overline{\mathcal{O}}_{0,\infty}(n_0)\,, \tag{4.31}$$

whenever the limit exists. We claim that for all $\mathcal{O} \in \mathcal{C}_{\mathbb{T}\,\mathrm{ainv}}$ the limit does exist and is given by the rhs of Eq. (4.13). To see this we return to (4.17) and swap the order of $K^{\mathcal{O}}$ and P:

$$\begin{aligned}
\mathcal{O}_{0,\infty}(n_0) &= \frac{\lambda_\beta(0)^{-x_\ell}}{\mathcal{P}_{-1/2}(n^\uparrow\cdot n_0)}\,(\mathbb{T}^{x_1}K^{\mathcal{O}}P)(n_0,n^\uparrow) \\
&= \frac{\lambda_\beta(0)^{x_1-x_\ell}}{\mathcal{P}_{-1/2}(n^\uparrow\cdot n_0)}\int d\Omega(n)\mathcal{P}_{-1/2}(n_0\cdot n)K^{\mathcal{O}}(n,n^\uparrow) \\
&+ \frac{\lambda_\beta(0)^{-x_\ell}}{\mathcal{P}_{-1/2}(n^\uparrow\cdot n_0)}\int d\Omega(n)\,\mathcal{T}_\beta(n_0\cdot n;x_1)[K^{\mathcal{O}},P](n,n^\uparrow)\,.
\end{aligned} \tag{4.32}$$

We now take the $\mathrm{SO}^\uparrow(2)$ average wrt n_0. In the first term the n_0 dependence then drops out by (A.12c) and produces the announced result. For the second term we use the defining bound (4.10) and distinguish between $x_1 = 0$ and $x_1 \neq 0$. In the first case a bound on the second term is $\lambda_\beta(0)^{-x_\ell}p_1 p(\xi_0)/\mathcal{P}_{-1/2}(\xi_0)$, which vanishes for $\xi_0 \to \infty$. For $x_1 \neq 0$ we bound the integral by $2\pi p_1 \int d\xi\, \overline{\mathcal{T}}_\beta(\xi_0,\xi;x)p(\xi)$, using the definition (4.10). To this integral we apply Lemma 2.2(iv) to get const$p(\xi_0)$, which again vanishes for $\xi_0 \to \infty$. This completes the proof of Proposition 4.3(ii). ∎

Let us add a number of comments on (4.13), (4.12). First one should note that the thermodynamic limit can be computed explicitly for all of $\mathcal{C}_{\mathbb{T}\,\mathrm{ainv}}$ without having to select 'fine-tuned' subsequences, i.e. without recourse to the Banach-Alaoglu theorem. Second one observes that translation invariance is restored in the thermodynamic limit even though for $\mathcal{O} \in \mathcal{C}_{\mathbb{T}\,\mathrm{ainv}}$ the finite volume expectations are not translation invariant. Third, just as for translation invariant observables the expectations (4.13), (4.12) will in general not be $\mathrm{SO}(1,2)$ invariant. An exception are observables in a subclass $\mathcal{C}^\uparrow_{\mathrm{ainv}} \subset \mathcal{C}_{\mathbb{T}\,\mathrm{ainv}}$ to be discussed below.

Just as $\mathcal{C}_{\mathbb{T}\,\mathrm{inv}}$ contained the $\mathrm{SO}(1,2)$ invariant observables $\mathcal{C}_{\mathrm{inv}}$ as special cases, here there is a subspace $\mathcal{C}_{\mathrm{ainv}}$ of observables which decay sufficiently fast to an $\mathrm{SO}(1,2)$ invariant one *after* averaging over $\mathrm{SO}^\uparrow(2)$. We denote the limiting observable by

$$\mathcal{O}_\infty(n_1,\ldots n_\ell) := \lim_{A\to\infty} \mathcal{O}(An_1,\ldots An_\ell)\,, \tag{4.33}$$

and specify the rate of approach to the limit below. Provided the limit exists it will automatically be $\mathrm{SO}(1,2)$ invariant. For example one can build a large class of $\mathcal{C}^\uparrow_p$ observables satisfying (4.33) by replacing in a function of $n_i\cdot n_j$ each $n_i\cdot n_j$ with $n_i\,\cdot$

$n_j\, f(n_i, n_j)$ or with $n_i \cdot n_j + f(n_i, n_j)$, for some $\mathrm{SO}^{\uparrow}(2)$ – but not $\mathrm{SO}(1,2)$ – invariant function f that goes to a constant in the limit. Note that the dependence on the invariant part may correspond to an unbounded function. In addition any dependence on the n_i^0 is allowed, constrained only by the requirement that the limit (4.33) exists. Of course $\mathcal{C}_{\rm ainv}$ contains the $\mathrm{SO}(1,2)$ invariant observables $\mathcal{C}_{\rm inv}$ as a proper subset. For observables depending only on one spin ($\ell = 1$) asymptotic translation invariance just reduces to the existence of the limit in (4.33), as it did for $\mathcal{C}_{\mathbb{T}\,\rm ainv}$ observables with $\ell = 1$.

For $\ell > 1$ we specify the rate of approach in which the limit in (4.33) is reached in terms of the kernels $K^{\mathcal{O}}$ as follows, thereby giving a technically precise version of Definition 3.3(ii):

Definition 4.5. $\mathcal{O} \in \mathcal{C}_p$ is called *asymptotically invariant*, $\mathcal{O} \in \mathcal{C}_{\rm ainv}$, iff after $\mathrm{SO}^{\uparrow}(2)$ averaging the associated kernel obeys

$$\begin{aligned}
&\Big|K^{\mathcal{O}}(n, n') - K^{\mathcal{O}_\infty}(n \cdot n')\Big| \leq p(n^{\uparrow}\!\cdot n)\, p(n^{\uparrow}\!\cdot n')\,, \quad \text{with} \\
&K^{\mathcal{O}_\infty}(n_1 \cdot n_\ell) := \lim_{A \to \infty} K^{\rho(A)\mathcal{O}}(n_1, n_\ell) = \lim_{A \to \infty} K^{\mathcal{O}}(A^{-1}n_1, A^{-1}n_\ell)\,,
\end{aligned}$$

where $p(\xi)$ is as in (4.10).

Note that in analogy with (4.11) this implies $\lim_{A\to\infty} \rho(A)[K^{\mathcal{O}}, \rho] = 0$, for $\mathcal{O} \in \mathcal{C}_{\rm ainv}$, which was used as the defining property in 3.3(ii). Further

$$\mathcal{C}_{\rm ainv} \subset \mathcal{C}_{\mathbb{T}\,\rm ainv}\,. \tag{4.34}$$

To see this one writes $[K^{\mathcal{O}}, \mathbb{T}^L] = [(K^{\mathcal{O}} - K^{\mathcal{O}_\infty}), \mathbb{T}^L] + [K^{\mathcal{O}_\infty}, \mathbb{T}^L]$. The second commutator vanishes because $\mathrm{SO}(1,2)$ invariant observables are translation invariant. The kernel of the first commutator is bounded in modulus by $\mathcal{P}_{-1/2}(n^{\uparrow} \cdot n)\, p(n^{\uparrow} \cdot n')\mathcal{T}_\beta(1; L)$. It follows that $[K^{\mathcal{O}}, P]$ satisfies (4.10), which verifies (4.34).

It follows that the formulae (4.13), (4.12) are valid also for observables in $\mathcal{C}_{\rm ainv}$. Moreover the kernel $\overline{K}^{\mathcal{O}}$ can in fact be replaced with the invariant limiting kernel $K^{\mathcal{O}_\infty}$.

Proposition 4.6.

(i) For a multi-point observable $\mathcal{O} \in \mathcal{C}_{\rm ainv}$, $\ell \geq 2$:

$$\langle \mathcal{O}(n_{x_1}, \ldots, n_{x_\ell}) \rangle_{\infty,\beta,\alpha} = \lambda_\beta(0)^{x_1 - x_\ell}\, 2\pi \int_1^\infty d\xi\, K^{\mathcal{O}_\infty}(\xi)\, \mathcal{P}_{-1/2}(\xi)\,. \tag{4.35}$$

(ii) On $\mathcal{C}^{\uparrow}_{\rm ainv}$ the expectation functional $\mathcal{O} \mapsto \langle \mathcal{O} \rangle_{\infty,\beta,\rm bc}$ is an invariant mean.

Proof. (i) We decompose $K^{\mathcal{O}}$ again as $K^{\mathcal{O}} = K^{\mathcal{O}_\infty} + (K^{\mathcal{O}} - K^{\mathcal{O}_\infty})$. For the invariant limiting kernel the manipulations proceed as for the translation invariant case and yield Eq. (4.13) with the indicated replacement. The average of the remainder

$K^{\mathcal{O}} - K^{\mathcal{O}_\infty}$ can be bounded by $Q(1; L + x_1)Q(n^\uparrow \cdot n_L; L - x_\ell)/\mathcal{T}_\beta(n^\uparrow \cdot n_L; 2L)$ with $Q(\xi; x) := 2\pi \int_1^\infty d\xi \overline{\mathcal{T}}_\beta(\xi, \xi'; x)p(\xi')$, for $x \in \mathbb{N}$. The bound vanishes in the limit $L \to \infty$.

(ii) This is a direct consequence of (4.35). ■

Remark 1. For later reference we note again that this reasoning remains valid if the lower boundary in the Q integrals was replaced with an arbitrarily large constant $\Lambda \gg 1$.

Remark 2. The reason why the expectation functionals do no not provide an invariant mean for all of $\mathcal{C}_{\rm ainv}$ is that the $\mathrm{SO}^\uparrow(2)$ averaging effected by the expectations does not commute with the $\mathrm{SO}(1,2)$ action. As a consequence observables in $\mathcal{C}_{\rm ainv} \setminus \mathcal{C}^\uparrow_{\rm ainv}$ will typically signal spontaneous symmetry breaking. See (3.26) and the examples in section 4.3. Likewise the hyperclustering (4.9) for $\mathrm{SO}(1,2)$ invariant observables trivially generalizes to the class $\mathcal{C}^\uparrow_{\rm ainv}$ but fails in general for $\mathcal{C}_{\rm ainv}$: if $\mathcal{A}, \mathcal{B} \in \mathcal{C}^\uparrow_{\rm ainv}$ have support as in the premise of (4.9) the limit of the products equals the product of the limits, i.e. $\overline{(\mathcal{A}\mathcal{B})}_\infty = \overline{\mathcal{A}}_\infty \overline{\mathcal{B}}_\infty$, and to the latter (4.9) applies. A counterexample to hyperclustering in $\mathcal{C}_{\rm ainv}$ will be given in section 4.3.

We can summarize these results by saying that the thermodynamic limit effectively projects $\mathcal{C}_{\mathbb{T}\,\rm ainv}$ onto $\mathcal{C}_{\mathbb{T}\,\rm inv}$ and $\mathcal{C}_{\rm ainv}$ onto $\mathcal{C}_{\rm inv}$, i.e. the top row in the diagram (3.9) is projected onto the bottom row. In the first case translation invariance emerges but $\mathrm{SO}(1,2)$ invariance is in general still absent, while in the second case, given $\mathrm{SO}^\uparrow(2)$ invariance as a 'seed', both properties emerge. The second result is more interesting because $\mathrm{SO}(1,2)$ is not amenable, so one could not 'by hand' switch to invariant states by group averaging of noninvariant ones. (Presumably this is still true if one adopts a distributional group averaging as in [34, 35].) Rather the thermodynamic limit itself defines a *partial invariant mean*, that is the subclass of (bounded as well as unbounded) observables $\mathcal{C}^\uparrow_{\rm ainv}$ gets averaged to yield a $\mathrm{SO}(1,2)$ invariant result. An invariant mean in the proper sense would do the same for all continuous bounded observables $\mathcal{C}^\uparrow_b$, but it cannot exist on general grounds.

4.3 The support of the functional measures

Here we discuss the results (4.12) and (4.35) in more detail. Both properties express a partial symmetry restoration and are due to a remarkable concentration (actually rather dilution) property of the underlying functional measures. Roughly speaking the measures have their support concentrated at configurations that are boosted from the origin by an amount growing at least powerlike with the number of sites; the measure of any bounded set of configurations goes to zero in the thermodynamic limit. The derivation of (4.12) given below explicitly makes use of this concentration property; the previous derivation of (4.35) did for technical reasons not explicitly rely on it. We shall explain later why the underlying concentration property is nevertheless visible in the derivation. In section 5.2 we shall also describe an alternative proof of (4.35) which links it explicitly to the concentration property of the 1-spin measures instrumental for (4.12). This concentration property is due to the large fluctuations present in $D = 1$,

which in compact models are the 'enforcers' of the Mermin-Wagner theorem, but which are here insufficient to restore the symmetry.

We begin with re-evaluating the thermodynamic limit for asymptotically translation invariant observables depending only on a single spin. Take some $\mathcal{O}(n_x) \in \mathcal{C}_{\mathbb{T}\,\mathrm{ainv}}$ (which for $\ell = 1$ coincides with $\mathcal{C}_{\mathrm{ainv}}$ by definition) depending on a single spin at site x only. By definition its $\mathrm{SO}^{\uparrow}(2)$ average has a limit as $n_x \to \infty$. We claim that this limiting value coincides with the thermodynamic limits of the $\mathcal{O}(n_x)$ expectation. The mechanism behind this is that the relevant measures have support 'mostly at infinity'. To make this precise, recall that in view of (3.13) and (3.17) the spin $n := n_x$ is for finite L distributed according to the probability measures

$$\begin{aligned} d\mu_{L,\beta,\alpha,\mathrm{av}}(n;x) &= \frac{\mathcal{T}_\beta(n^\uparrow \cdot n; L+x)\overline{\mathcal{T}}_\beta(n^\uparrow \cdot n, \mathrm{ch}\alpha; L-x)}{\mathcal{T}_\beta(\mathrm{ch}\alpha; 2L)}\, d\Omega(n)\,, \\ &\quad \text{for twisted bc} \\ d\mu_{L,\beta,\mathrm{free}}(n;x) &= \mathcal{T}_\beta(n^\uparrow \cdot n; L+x)\, d\Omega(n)\,, \\ &\quad \text{for free bc}\,. \end{aligned} \tag{4.36}$$

In the first case $\overline{\mathcal{T}}_\beta$ is defined as in (2.28). From Eqs. (2.27), (2.36) and (2.31) one sees that the densities multiplying $d\Omega$ behave for large L as

$$\begin{aligned} &L^{-3/2}\,\big(\mathcal{P}_{-1/2}(n^\uparrow \cdot n)\big)^2\,, \qquad &&\text{for twisted bc,} \\ &\lambda_\beta(0)^{L+x}\, L^{-3/2}\, \mathcal{P}_{-1/2}(n^\uparrow \cdot n)\,, \qquad &&\text{for free bc}\,; \end{aligned} \tag{4.37}$$

(the approach to this asymptotic form is, however, very nonuniform in n^0, as can be seen from Eqs. (2.35),(2.38)). In particular the dependence on $\mathrm{ch}\alpha$ drops out in the first case. Both expressions in (4.36),(4.37) vanish pointwise in the limit but are not integrable. This implies

Lemma 4.7. *For any bounded subset $M \subset \mathbb{H}$ and for twisted as well as free bc*

$$\lim_{L\to\infty} \int_M d\mu_{L,\mathrm{bc}} = 0\ , \tag{4.38}$$

where $d\mu_{L,\mathrm{bc}}$ stands for either of the measures in (4.36).

As a consequence these measures do not have a limit as $L \to \infty$; they 'spread out' over $\mathbb{H}$ (though not evenly); Lemma 4.7 may be interpreted as saying that the measure is getting concentrated more and more near infinity.

The measures $d\mu_{L,\beta,\mathrm{bc}}$ form a sequence of bounded, normalized linear functionals ('states') on the space $\mathcal{C}_b$. By the theorem of Banach-Alaoglu [30] there is therefore a subsequence convergent to such a functional – a so-called 'mean'; see e.g. [4]. Because $\mathrm{SO}(1,2)$ is not amenable, this mean cannot be invariant. We will give below explicit examples of elements of $\mathcal{C}_b$ that show this non-invariance, i.e. spontaneous symmetry breaking. However 1-spin observables invariant under $\mathrm{SO}^{\uparrow}(2)$ still have a *unique* thermodynamic limit,

which is independent of x and β, see Proposition 4.3(i). In view of Lemma 4.7 this expresses the fact that the thermodynamic limit effectively projects a one spin observable onto the 'boundary at infinity' of the hyperbolic plane; for $\mathrm{SO}^{\uparrow}(2)$ invariant functions we may use the one-point compactification of $\mathbb{H}$ so that there is only one such boundary point at infinity.

It is also instructive to estimate the size of the 'cup' in the hyperboloid whose contribution to the functional integral is negligible. We integrate the observable under consideration with the pointwise vanishing density in (4.37) over the compact domain $\{n \in \mathbb{H} | \, n^{\uparrow} \cdot n \leq \Lambda(L)\}$. Demanding that the contribution of this domain still vanishes in the limit $L \to \infty$ constrains the permitted growth of $\Lambda(L)$ with L. For twisted bc the relevant integral is $\int^{\Lambda(L)} d\xi \mathcal{P}_{-1/2}(\xi)^2 \sim \int^{\Lambda(L)} d \ln \xi (\ln \xi)^2 \sim (\ln \Lambda(L))^3$, using (4.37) and the asymptotics in (A.13). Thus any $\Lambda(L)$ satisfying $\ln \Lambda(L) = o(L^{1/2})$ will still give a contribution vanishing in the limit $L \to \infty$. For free bc the relevant integral is $\int^{\Lambda(L)} d\xi \mathcal{P}_{-1/2}(\xi) \sim \Lambda(L) \ln \Lambda(L)$. Thus any growth $\Lambda(L) = o(L^{3/2}/\ln^2 L)$ is allowed.

To conclude our discussion of 1-point functions let us consider some examples. A simple example of a bounded $\mathrm{SO}^{\uparrow}(2)$ invariant observable is $\mathcal{O}(n) = \tanh(n^{\uparrow} \cdot n)$. Then (4.12) gives 1 for the thermodynamic limit of its expectation, which in particular is $\mathrm{SO}(1,2)$ invariant. The spin field n_x^a itself is neither bounded nor $\mathrm{SO}^{\uparrow}(2)$ invariant. However by the $\mathrm{SO}^{\uparrow}(2)$ invariance of the measures (4.36) one has

$$\langle n_x^a \rangle_{L,\beta,\mathrm{bc}} \;=\; \delta^{a0} \langle n_{-L} \cdot n_x \rangle_{L,\beta,\mathrm{bc}} \,, \tag{4.39}$$

leaving only the $\mathrm{SO}^{\uparrow}(2)$ invariant part $\mathcal{O}(n_x) = n^{\uparrow} \cdot n_x = n_x^0$ to study. Slightly generalizing the above discussion it follows that the n_x^0 expectation with both twisted periodic and free bc diverges for $L \to \infty$: for any constant Λ, the measure of the (compact) subset of $\mathbb{H}$ where $|n_x^0| \leq \Lambda$ goes to 0; since the total weight of the measure is always 1, the expectation value will eventually be larger than $\Lambda(1-\epsilon)$ for any $\epsilon > 0$. For the rhs in (4.39) this is also illustrated by the numerical results for the 2-point functions shown in Fig. 2. In both of these examples the limit is $\mathrm{SO}(1,2)$ invariant and does not signal spontaneous symmetry breaking.

As seen before the computation of the thermodynamic limit for multi-point observables can be reduced to that of 1-point functions. Nevertheless it is instructive to outline the origin of the concentration property also for the multi-point measures. For simplicity we restrict attention to twisted bc. The counterpart of the normalized measures (4.36) for $\ell > 1$ are (after integrating out $n_{x_2}, \ldots n_{x_{\ell-1}}$)

$$d\mu_{L,\beta,\alpha}(n_1, n_\ell; x_1, x_\ell) = \tag{4.40}$$
$$\frac{\mathcal{T}_\beta(n^{\uparrow} \cdot n_1; L + x_1) \mathcal{T}_\beta(n_1 \cdot n_\ell; x_\ell - x_1) \overline{\mathcal{T}}_\beta(\mathrm{ch}\alpha, n^{\uparrow} \cdot n_\ell; L - x_\ell)}{\mathcal{T}_\beta(\mathrm{ch}\alpha; 2L)} \, d\Omega(n_1) d\Omega(n_\ell) \,.$$

The finite volume expectation of some observable $\mathcal{O}$ can be written in terms of these measures as

$$\langle \mathcal{O} \rangle_{L,\beta,\mathrm{bc}} = \int d\mu_{L,\beta,\mathrm{bc}}(n_1, n_\ell) \, \frac{K^{\mathcal{O}}(n_1, n_\ell)}{\mathcal{T}_\beta(n_1 \cdot n_\ell; x_\ell - x_1)} \,. \tag{4.41}$$

The asymptotics of the density in (4.40) is

$$\lambda_\beta(0)^{x_1-x_\ell}\,\mathcal{P}_{-1/2}(n^\uparrow\cdot n_1)\,\mathcal{T}_\beta(n_1\cdot n_\ell;x_\ell-x_1)\mathcal{P}_{-1/2}(n^\uparrow\cdot n_\ell)\;L^{-3/2}\,. \tag{4.42}$$

This density vanishes pointwise as $L\to\infty$ and is integrable wrt one but not wrt both variables. As before the limit of the measures therefore only exists as a mean.

The concentration property ensued by (4.42) is however more subtle than for the 1-point measures. This is because invariant combinations like $n_1\cdot n_\ell$ contribute even for highly boosted individual n_1 and n_ℓ. Conditions like (asymptotic) translation invariance or (asymptotic) SO$(1,2)$ invariance allow one to isolate the invariant contribution by swapping the order of $K^{\mathcal{O}}$ and $\mathbb{T}^{L+x_1}$ while implying that the commutator does not contribute to the invariant part. In order to illustrate the mechanism we set

$$k(n^\uparrow\!\cdot n_1):=\sup_{n_\ell}\frac{\overline{K}^{\mathcal{O}}(n^\uparrow\cdot n_1,n^\uparrow\cdot n_\ell)}{\mathcal{T}_\beta(n_1\cdot n_\ell;x_\ell-x_1)}\,. \tag{4.43}$$

Clearly $|k(\xi_1)|\leq\|\mathcal{O}\|$. If $\mathcal{O}$ and hence $K^{\mathcal{O}}$ is SO$(1,2)$ invariant, $k(\xi_1)$ equals a constant. If $K^{\mathcal{O}}$ does not contain a SO$(1,2)$ part the function $k(\xi_1)$ vanishes for $\xi_1\to\infty$. Then

$$\begin{aligned}
&\int_{n^\uparrow\cdot n_1<\Lambda_1} d\mu_{L,\beta,\alpha}(n_1,n_\ell;x_1,x_\ell)\frac{K^{\mathcal{O}}(n_1,n_\ell)}{\mathcal{T}_\beta(n_1\cdot n_\ell;x_\ell-x_1)}\\
&\quad\leq\int_1^{\Lambda_1}d\xi\,k(\xi)\,\frac{\mathcal{T}_\beta(\xi;L+x_1)\overline{\mathcal{T}}_\beta(\xi,n^\uparrow\cdot n_L;L-x_1)}{\mathcal{T}_\beta(n^\uparrow\cdot n_L;2L)}\leq\frac{\mathcal{T}_\beta(1;L+x_1)\mathcal{T}_\beta(1;L-x_1)}{\mathcal{T}_\beta(n^\uparrow\cdot n_L;2L)}\\
&\quad\times\mathcal{P}_{-1/2}(n^\uparrow\!\cdot n_L)\int_1^{\Lambda_1}d\xi\,k(\xi)\,\mathcal{P}_{-1/2}(\xi)^2\,E\Big(\frac{\ln\xi}{\sqrt{L+x_1}}\Big)\,.
\end{aligned} \tag{4.44}$$

In the last step we used the SO$^\uparrow(2)$ average of the bound in Lemma 2.2(iii) and (2.38). The estimates in (4.44) capture the qualitative features of the concentration phenomenon. There are two cases to consider: (i) $K^{\mathcal{O}}$ does not contain an SO$(1,2)$ invariant part, in which case $k(\xi_1)\to0$ as $\xi_1\to\infty$. Using the first bound in (4.44) and the argument used for 1-point functions one sees that its $L\to\infty$ limit is given by the $\xi\to\infty$ limit of $k(\xi)$ and thus vanishes, both for finite Λ_1 and for $\Lambda_1\to\infty$. (ii) $K^{\mathcal{O}}$ does contain an SO$(1,2)$ invariant part, in which case $\lim_{\xi\to\infty}k(\xi)\neq0$. In this case it is instructive to estimate the size of the 'cup' in the n_1 hyperboloid that does not contribute significantly to the average as L becomes large. To this end we use the second bound in (4.44) and note that for fixed L one can take the $\Lambda_1\to\infty$ limit at the price that the integral scales like $L^{3/2}$ for large L. In other words one simple recovers the normalizibility of the measures in the regime $\ln\Lambda_1\gg\sqrt{L}$. On the other hand for $\ln\Lambda_1\ll\sqrt{L}$ the integral scales like $(\ln\Lambda_1/\sqrt{L})^2$. In particular one can allow Λ_1 to grow with L according to

$$\ln\Lambda_1(L)=o(L^{1/2})\,, \tag{4.45}$$

and still have the bound in (4.44) vanish for $L \to \infty$. (Note that this conclusion only depends on the simple bound Lemma 2.2 (iii) and not on (2.38).) The intermediate regime can also be analyzed; a typical case is $\ln \Lambda_1 = L^q$ with $q > 1/2$, for which the integral in (4.44) approaches a finite but nonzero constant as $L \to \infty$. The upshot is that in the original (n_1, n_ℓ) integral over $\mathbb{H} \times \mathbb{H}$ only the region $n^\uparrow \cdot n_1 \geq \Lambda_1(L)$, with $\Lambda_1(L)$ as in (4.45), contributes significantly to the result for the average as L becomes large.

For free bc the analysis is similar, except that the change in the rate of decay also involves powers of $\lambda_\beta(0)$. We omit the details and simply state that one can likewise allow the cutoff Λ_1 to grow at least powerlike in L, without affecting the limit formulas.

4.4 Examples

We begin with some examples where a finite thermodynamic limit does not necessarily exist, like for the components of the spin field or of the Noether current.

The individual components of the energy observable $E^a_{L,\beta,\mathrm{bc}} := \langle n^a_x\, n^a_{x+1}\rangle_{L,\beta,\mathrm{bc}}$, $a = 0, 1, 2$, can be shown to diverge for $L \to \infty$ by an argument similar to the one used in section 4.2. On the other hand the invariant combination $(E^0 - 2E^1)_{L,\beta,\mathrm{bc}}$ has a finite limit given by (4.8). Next consider the Noether current $J^a_x = \beta(n_x \times n_{x+1})^a$, where $n \times n'$ denotes the $\mathrm{SO}(1,2)$ invariant vector product of $n, n' \in \mathbb{H}$. (Explicitly $(n \times m)^a = \eta^{aa'}\epsilon_{a'bc} n^b m^c$, with ϵ_{abc} totally antisymmetric and $\epsilon_{012} = 1$.) For the current two-point function one finds

$$\begin{aligned} &\langle J^0_x J^0_y\rangle_{L,\beta,\mathrm{bc}} = 0\,, \quad \text{for } x < y\,, \\ &\langle J^1_x J^1_y\rangle_{L,\beta,\mathrm{bc}} = \langle J^2_x J^2_y\rangle_{L,\beta,\mathrm{bc}} = -\frac{1}{2}\langle J_x \cdot J_y\rangle_{L,\beta,\mathrm{bc}}\,, \quad \text{for } x < y\,, \end{aligned} \tag{4.46}$$

so that all components have a finite $L \to \infty$ limit. The first equation is a special case of the more general result

$$\langle J^0_{x_1}\, \mathcal{O}(n_{x_2}, \ldots, n_{x_\ell})\rangle_{L,\beta,\mathrm{bc}} = 0\,, \quad \text{for } x_1 < x_2 < \ldots < x_\ell\,, \tag{4.47}$$

which is obtained by specializing the general formulas (3.12), (3.16) and then using

$$\frac{\partial}{\partial \varphi_x} \mathcal{T}_\beta(n_x \cdot n_{x+1}; 1) = -J^0_x\, \mathcal{T}_\beta(n_x \cdot n_{x+1}; 1)\,, \tag{4.48}$$

where $n_x \cdot n_{x+1} = \xi_x \xi_{x+1} - \sqrt{\xi_x^2 - 1}\sqrt{\xi_{x+1}^2 - 1}\cos(\varphi_x - \varphi_{x+1})$. Since J^0_x is essentially the Noether charge generating infinitesimal $\mathrm{SO}^\uparrow(2)$ rotations (see below) Eq. (4.47) expresses the $\mathrm{SO}^\uparrow(2)$ invariance of the 'ground states' 1 and $\psi_\uparrow(n)$, respectively. Conversely, the fact that correlators involving J^1_x, J^2_x are non-zero is yet another manifestation of the $\mathrm{SO}(1,2)$ symmetry breaking.

Ward identities expressing the invariance of the measure and of the action can be derived along the familiar lines. For example one has

$$\langle (J_x^a - J_{x+1}^a)\, n_y^b \rangle_{L,\beta,\mathrm{bc}} + \delta_{x,y}\, \langle (t^a n_y)^b \rangle_{L,\beta,\mathrm{bc}}\,, \tag{4.49}$$

with $(t^a)_c^d = -\eta^{aa'}\eta^{dd'}\epsilon_{a'd'c}$. Replacing n_y^b with a generic (non-invariant) observable $\mathcal{O}(n_{x_1},\ldots,n_{x_\ell})$ a similar identity arises where the correlator with $J_x^a - J_{x+1}^a$ produces a sum of contact terms. As is clear from (4.49) these linear Ward identities will in general not have a non-boring thermodynamic limit. In particular no conflict, even in spirit, with Coleman's theorem [36] arises.

Ward identities where the current enters nonlinearly can likewise be derived but are hampered by the fact that the 'response' are in general functions which fail to be translation invariant. In 2 or more dimensions a useful quadratic Ward identity can be derived which relates the components of the longitudinal part of the current-current correlator to the energies E^a; see [37]. In one dimension only the longitudinal part exists and only the $\mathrm{SO}(1,2)$ invariant – and hence translation invariant – combination of these component Ward identities is useful. It reads

$$\langle J_p \cdot J_{-p} \rangle_{L,\beta,\mathrm{bc}} + 2\beta(E^0 - 2E^1)_{L,\beta,\mathrm{bc}} = 0\,, \quad \forall p \neq 0\,, \tag{4.50}$$

where $J_p^a = \sum_x e^{-ipx} J_x^a$, with $p = 2\pi n/(2L+1)$, $n = 0,\ldots,2L$.

Next we consider some examples of asymptotically translation invariant observables. They also serve to highlight the significance of the $\mathrm{SO}^\uparrow(2)$ averaging in the definition of the algebras in (3.9). Recall that $\mathcal{C}_{\mathrm{ainv}} = \{\mathcal{O} \in \mathcal{C}_p \,|\, \mathrm{SO}^\uparrow(2) \text{ average lies in } \mathcal{C}_{\mathrm{ainv}}^\uparrow\}$. The point here is that in general $\mathcal{O}$ and $\rho(A)\mathcal{O}$, $A \in \mathrm{SO}(1,2)$, will have *different* $\mathrm{SO}^\uparrow(2)$ invariant images in $\mathcal{C}_{\mathrm{ainv}}^\uparrow$. The elements of $\mathcal{C}_{\mathrm{ainv}} \setminus \mathcal{C}_{\mathrm{ainv}}^\uparrow$ will therefore typically signal spontaneous symmetry breaking although by section 3.3 they get effectively projected back into $\mathcal{C}_{\mathrm{ainv}}^\uparrow$.

An instructive example of such a 'symmetry breaking observable' in $\mathcal{C}_{\mathrm{ainv}}$ arises as follows: given a spacelike unit vector $e = (\sqrt{q^2-1}, q\sin\gamma, q\cos\gamma)$ we define

$$\begin{aligned} T_e(n) &:= \tanh(n\cdot e) \in \mathcal{C}_{\mathrm{ainv}}\,, \\ \overline{T}_q(\xi) &:= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi\, \tanh\Big(\xi\sqrt{q^2-1} - q\sqrt{\xi^2-1}\,\cos\varphi\Big) \in \mathcal{C}_{\mathrm{ainv}}^\uparrow\,. \end{aligned} \tag{4.51}$$

The observable $T_e(n)$ indeed enjoys the property (4.34): after $\mathrm{SO}^\uparrow(2)$ averaging it has a unique limit $\overline{T}_q(\infty)$, which can be obtained by acting with a sequence of $\mathrm{SO}(1,2)$ transformations going to infinity. This limit does not depend on n any more, so in a trivial sense it is an invariant function of the spins. It does, however, depend on e or rather on the scalar product $n^\uparrow \cdot e$ and is therefore not invariant under the action of $\mathrm{SO}(1,2)$ on the original observable.

Spontaneous symmetry breaking is shown by the following

Proposition 4.8. *For all bc considered*

$$\langle T_e(n_x)\rangle_{\infty,\beta,\mathrm{bc}} = \overline{T}_q(\infty) = 1 - \frac{2}{\pi}\arccos\sqrt{1-q^{-2}}\ ; \tag{4.52}$$

this expectation value is manifestly not invariant under $\mathrm{SO}(1,2)$*: for a general* $A \in \mathrm{SO}(1,2)$ *one has* $\langle T_e(n_x)\rangle_{\infty,\beta,\mathrm{bc}} \neq \langle T_e(An_x)\rangle_{\infty,\beta,\mathrm{bc}}$.

Proof. We use the fact that in finite volume expectations we may replace $T_e(n_x)$ by it average over $\mathrm{SO}^{\uparrow}(2)$ rotations $\overline{T}_q(n_x^0)$. The argument of the tanh, i.e. $\alpha_\xi(\varphi) := \xi(\sqrt{q^2-1} - q\sqrt{1-\xi^{-2}}\cos\varphi)$, then has its minimum at $\varphi = 0$ and its maximum at $\varphi = \pm\pi$. Because e is spacelike, there is a $\xi_0(q)$ such that for all $\xi > \xi_0(q)$ the minimum $\alpha_\xi(0)$ is negative, the maximum $\alpha_\xi(\pm\pi)$ is positive and there are two zeros at $\varphi = \pm\arccos[(1-q^{-2})/(1-\xi^{-2})]^{1/2}$ whose modulus converges to $\varphi_0 := \arccos(1-q^{-2})^{1/2}$. This implies that

$$\lim_{\xi\to\infty}\tanh\alpha_\xi(\varphi) = \begin{cases} 1, & |\varphi| > \varphi_0 \\ -1, & |\varphi| < \varphi_0 \\ 0, & |\varphi| = \varphi_0. \end{cases} \tag{4.53}$$

By the dominated convergence theorem we can pull the limit $\xi \to \infty$ under the integral for the φ averaging and obtain

$$\lim_{\xi\to\infty}\overline{T}_q(\xi) = 1 - \frac{2}{\pi}\arccos\sqrt{1-q^{-2}}\,. \tag{4.54}$$

The result then follows from Eq. (4.12). ∎

A large class of observables in $\mathcal{C}_{\mathrm{ainv}}$ can now be built by algebraic operations. Of course sums and products of $T_e(n)$ at the same or different sites will lie in $\mathcal{C}_{\mathrm{ainv}}$, but so will be algebraic combinations built from elements of $\mathcal{C}_{\mathrm{inv}}$. The crucial difference to $\mathcal{C}^{\uparrow}_{\mathrm{ainv}}$ is that hyperclustering and even ordinary clustering will now *fail* in general. This is because ‘$\mathrm{SO}^{\uparrow}(2)$ averaging’ and ‘taking the $A \to \infty$ limit’ in (4.33) are noncommuting operations in general. A simple example is given by the product of two tanh-observables (4.51), where

$$1 = \langle T_e(n)^2\rangle_{\infty,\beta,\mathrm{bc}} \neq \langle T_e(n)\rangle^2_{\infty,\beta,\mathrm{bc}} = \overline{T}_q(\infty)^2\,, \tag{4.55}$$

from (4.53) and (4.54).

So we have so far found observables that show hyperclustering and others that do not cluster at all. Observables showing ordinary (exponential or power-like) clustering presumably also exist in the large space $\mathcal{C}_b$, but it is more difficult to find explicit examples.

5. Reconstruction of a Hilbert space and transfer operator

The Osterwalder-Schrader type reconstruction allows to reconstruct a Hilbert space and a transfer matrix from expectation values satisfying reflection positivity as well as translation invariance. The original expectation values are recovered as expectations in a genuine, i.e. *normalizable* ground state vector. This construction is well documented in the literature [38, 39, 40], but in our case there are peculiarities and surprises. For this reason we describe in some detail how the construction works here.

First there is a rather harmless complication: reflection positivity for reflections both in lattice sites and in midpoints between lattice points is equivalent to positivity of the transfer operator; as we found in the beginning, however, this does not hold in our case. But we still have reflection positivity for reflection in lattice points, at least if we take the thermodynamic limit with periodic bc, and this is enough for the reconstruction of the Hilbert space and a positive two-step transfer matrix.

There is a much more serious complication: as stated above, the reconstruction produces a ground state in the proper sense, whereas we know that the original transfer matrix $\mathbb{T}$ on $L^2(\mathbb{H})$ does not have such a ground state. So it is unavoidable that there is some discrepancy between the reconstructed quantum mechanics and the one we started from. This mismatch is also related to the fact that our expectation functional in the infinite volume is not given by a measure, but only a mean on the configuration space.

In this section we consider periodic bc exclusively and denote the expectation functional (the state) $\langle\ .\ \rangle_{L,\beta,0}$ in Eq. (3.12) by $\omega_L(\,.\,)$. A reconstruction in the usual sense won't work for twisted or free bc because the $x \geq 0$ and the $x \leq 0$ halves of the chain have to enter symmetrically. For the algebra we take $\mathcal{C}_b$ in order to have the usual concept of a state available. For $\mathcal{C}_b \cap \mathcal{C}_{\mathbb{T}\,\mathrm{ainv}}$ we saw before that the thermodynamic limit is explicitly computable and translation invariant. For the rest of $\mathcal{C}_b$ a thermodynamic limit exists likewise, though it may be necessary to select subsequences and to average over translations in order to have it translation invariant. We denote such a weak limiting state by $\omega_\infty(\,.\,) = w - \lim_{L\to\infty} \omega_L(\,.\,)$.

We denote by $\mathcal{C}_+$ ($\mathcal{C}_-$) the subalgebra of bounded observables $\mathcal{C}_b$ depending only on the spins n_x with $x \geq 0$, ($x \leq 0$) and $\mathcal{C}_0 = \mathcal{C}_+ \cap \mathcal{C}_-$. Our chain admits a reflection $x \to -x$ and we introduce an antilinear time reflection ϑ acting on on $\mathcal{C}_b$ by replacing any function $\mathcal{O}$ by the same function of the reflected arguments and taking the complex conjugate:

$$(\vartheta\mathcal{O})(n_{-x_{\ell-1}}, \ldots, n_{-x_0}) = \mathcal{O}(n_{-x_0}, \ldots, n_{-x_{\ell-1}})^* \,, \quad x_0 < \ldots < x_{\ell-1} \,, \tag{5.1}$$

where the asterisk denotes complex conjugation. To interpret this formula correctly note that on the lhs we have written the observable $\vartheta\mathcal{O}$ in the customary form as a function of the spins on which it actually depends, in the order of increasing indices. On the rhs $\mathcal{O}$ is to be read as a function of ℓ spins, with the displayed arguments now appearing in the order of decreasing indices. For example $\mathcal{O} = n_1 \cdot n_3 + c\, n^\uparrow \cdot n_3$ gives

$\vartheta\mathcal{O} = n_{-1}\cdot n_{-3} + c^*\, n^\uparrow \cdot n_{-3}$. We discuss the reconstruction first for a finite and then for an infinite chain.

5.1 Finite chains

Recall that we adopt untwisted periodic bc, $n_{-L} = n_L = n^\uparrow$, and consider a chain of total length $2L+1$. We begin by assigning to each $\mathcal{O}\in\mathcal{C}_+$ an element $\mathcal{O}_{0,L}\in\mathcal{C}_0$ with the same expectation value via

$$\begin{aligned}
\mathcal{O}_{0,L}(n_0) &:= \int \prod_{i=1}^{\ell} d\Omega(n_i)\mathcal{O}(n_1,\ldots,n_\ell)\prod_{i=1}^{\ell}\mathcal{T}_\beta(n_{i-1}\cdot n_i; x_i - x_{i-1})\frac{\mathcal{T}_\beta(n_\ell\cdot n^\uparrow; L-x_\ell)}{\mathcal{T}_\beta(n_0\cdot n^\uparrow; L)}\\
&= \int d\Omega(n)\,(\mathbb{T}^{x_1}K^{\mathcal{O}})(n_0,n)\frac{\mathcal{T}_\beta(n^\uparrow\cdot n; L-x_\ell)}{\mathcal{T}_\beta(n^\uparrow\cdot n_0; L)}\,,
\end{aligned} \tag{5.2}$$

where $x_0 = 0$ and the first transfer matrix is to be interpreted as the identity operator if $x_1 = 0$. Note the properties

$$\begin{aligned}
&|\mathcal{O}_{0,L}(n)| \leq \|\mathcal{O}\|\,, \qquad (1\!\!1)_{0,L}(n) = 1\!\!1\,,\\
&(\rho(A)\mathcal{O})_{0,L}(n) = \mathcal{O}_{0,L}(A^{-1}n)\Big|_{n^\uparrow\mapsto A^{-1}n^\uparrow}
\end{aligned} \tag{5.3}$$

where we denote by $1\!\!1$ the unit element of $\mathcal{C}_+$. Further we set

$$\psi^{\mathcal{O}}(n) := \mathcal{O}_{0,L}(n)\,\frac{\mathcal{T}_\beta(n\cdot n^\uparrow; L)}{\sqrt{\mathcal{T}_\beta(1;2L)}}\,. \tag{5.4}$$

The expressions (5.2) and (5.4) are designed such that

$$\omega_L([\vartheta\mathcal{O}]\,\mathcal{O}) = \frac{1}{\mathcal{T}_\beta(1;2L)}\int d\Omega(n)\,|\mathcal{O}_{0,L}(n)|^2\,\mathcal{T}_\beta(n\cdot n^\uparrow; L)^2 = \int d\Omega(n)\,|\psi^{\mathcal{O}}(n)|^2\,, \tag{5.5}$$

holds, as one can verify from (3.12). In particular reflection positivity

$$\omega_L([\vartheta\mathcal{O}]\mathcal{O}) \geq 0\,,\quad \forall\,\mathcal{O}\in\mathcal{C}_+\,, \tag{5.6}$$

is manifest.

With these preparations at hand the reconstruction of the Hilbert space $\mathcal{H}_L$ for a finite chain works as usual: a positive semidefinite scalar product is introduced on $\mathcal{C}_+$ by

$$(\mathcal{A},\mathcal{B})_L := \omega_L([\vartheta\mathcal{B}]\,\mathcal{A})\,; \tag{5.7}$$

there will be a nontrivial null space $\mathcal{N}$ of elements with $\omega_L([\vartheta\mathcal{O}]\,\mathcal{O}) = 0$. The Hilbert space $\mathcal{H}_L$ is then the completion of the quotient space $\mathcal{C}_+/\mathcal{N}$ with respect to the norm induced by ω_L. The necessity to divide out $\mathcal{N}$ becomes clear if one notices that for any $\mathcal{O} \in \mathcal{C}_+$ one can find a unique element $\mathcal{O}_{0,L} \in \mathcal{C}_0$ such that $\mathcal{O} - \mathcal{O}_{0,L} \in \mathcal{N}$, namely just the one given in (5.2). The uniqueness follows from (5.5), which implies $\mathcal{C}_0 \cap \mathcal{N} = \{0\}$. Note that the OS norm for $\mathcal{O}$ coincides with the L^2-norm for $\psi^{\mathcal{O}}$.

The above construction makes it manifest that for a finite chain there is a natural isometry between the reconstructed Hilbert space $\mathcal{H}_L$ and the original $L^2(\mathbb{H})$: $\mathcal{H}_L$ turned out to be the completion of $\mathcal{C}_0$ with respect to the norm induced by (5.7), i.e. $\mathcal{H}_L = \overline{\mathcal{C}}_0$. Note that although $\mathcal{C}_0$ is the universal L-independent space of bounded continuous functions on $\mathbb{H}$, its completion with respect to $(\ ,\ )_L$ depends on L. Of course the L-dependence is of a rather trivial nature in that by (5.4) the map

$$V_L : \mathcal{H}_L \;\longrightarrow\; L^2(\mathbb{H}) \,, \qquad (V_L\psi)(n) = \psi(n)\,\frac{\mathcal{T}_\beta(n\cdot n^\uparrow; L)}{\sqrt{\mathcal{T}_\beta(1; 2L)}} \,, \tag{5.8}$$

defines an isometry between Hilbert spaces. Alternatively $\mathcal{H}_L$ could be regarded as the preimage of $L^2(\mathbb{H})$ with respect to V_L. It is worth noting that $\mathcal{H}_L$ is by itself a commutative C^*-algebra, so the reconstruction of the Hilbert space can be considered as an instance of the well-known Gel'fand-Naĭmark-Segal reconstruction, see e.g. [1, 41]. To sum up, for a finite chain the original Hilbert space $L^2(\mathbb{H})$ and the reconstructed one $\mathcal{H}_L$ can really be identified.

Unsurprisingly, for a finite chain $\mathcal{H}_L$ also carries a unitary representation ρ_L of SO(2, 1) (spontaneous symmetry breaking can only arise in the thermodynamic limit); it is obtained simply by conjugating the representation ρ with V_L:

$$\rho_L = V_L^{-1}\rho V_L \,. \tag{5.9}$$

Explicitly for $A \in \mathrm{SO}(2,1)$ and $\psi \in \mathcal{C}_0$ this gives

$$(\rho_L(A)\psi)(n) = \psi(A^{-1}n)\frac{\mathcal{T}_\beta(A^{-1}n\cdot n^\uparrow; L)}{\mathcal{T}_\beta(n\cdot n^\uparrow; L)} \,. \tag{5.10}$$

For $\mathcal{O} \in \mathcal{C}_+$ we define

$$(\rho_L(A)\mathcal{O})(n_{x_0},\ldots,n_{x_{\ell-1}}) = \mathcal{O}(A^{-1}n_{x_0},\ldots,A^{-1}n_{x_{\ell-1}})\frac{\mathcal{T}_\beta(A^{-1}n_{x_{\ell-1}}\cdot n^\uparrow; L - x_{\ell-1})}{\mathcal{T}_\beta(n_{x_{\ell-1}}\cdot n^\uparrow; L - x_{\ell-1})} \,, \tag{5.11}$$

which is compatible with (5.10) and induces it via (5.2) in that $(\rho_L(A)\mathcal{O})_{0,L}(n) = (\rho_L(A)\mathcal{O}_{0,L})(n)$, for all $A \in \mathrm{SO}(1,2)$. This also ensures that ρ_L maps elements $\mathcal{O} - \mathcal{O}_{0,L}$ of $\mathcal{N}$ onto other elements of zero norm. The rhs of (5.10), (5.11) is in general no longer

a bounded function of n because the asymptotics of $\mathcal{T}_\beta(A^{-1}n\cdot n^\uparrow;L)$ and $\mathcal{T}_\beta(n\cdot n^\uparrow;L)$ do not match, but it is of course still an element of $\mathcal{H}_L$ with the same norm as ψ. Likewise by (5.10), (5.11) the completion $\mathcal{N}_L$ of $\mathcal{N}$ wrt ω_L is mapped onto itself under ρ_L.

In preparation of the thermodynamic limit let us consider the action of ρ_L on the function $1\!\!1$ (an approximate ground state for large L):

$$(\rho_L(A)1\!\!1)(n) \;=\; \frac{\mathcal{T}_\beta(A^{-1}n\cdot n^\uparrow;L)}{\mathcal{T}_\beta(n\cdot n^\uparrow;L)}\,. \tag{5.12}$$

The scalar product

$$\begin{aligned}(\rho_L(A)1\!\!1,\rho_L(B)1\!\!1)_L \;&=\; \frac{1}{\mathcal{T}_\beta(1;2L)}\int d\Omega(n)\mathcal{T}_\beta(n\cdot An^\uparrow;L)\mathcal{T}_\beta(n\cdot Bn^\uparrow;L)\\ &=\; \frac{\mathcal{T}_\beta(An^\uparrow\cdot Bn^\uparrow;2L)}{\mathcal{T}_\beta(1;2L)}\,,\end{aligned} \tag{5.13}$$

then has the finite and nonzero limit $\mathcal{P}_{-1/2}(An^\uparrow\!\cdot\! Bn^\uparrow)$, as $L\to\infty$.

For finite L the state $\omega_L(\,\cdot\,)$ will not be translation invariant outside the subalgebra $\mathcal{C}_{\mathbb{T}\,\mathrm{inv}}\cap\mathcal{C}_+$. As a consequence there is no reconstructed transfer matrix for finite L. Conversely this provides an intrinsic reason to consider the reconstruction based on the expectations of the infinite chain.

5.2 Thermodynamic limit

Let us thus turn to the thermodynamic limit $\omega_\infty = w-\lim_{L\to\infty}\omega_L$. Reflection positivity remains true in this limit; so one can still define a scalar product and a null space $\mathcal{N}$ as for the finite chain. As before a Hilbert space $\mathcal{H}_{OS}$ can be constructed as the completion of $\mathcal{C}_+/\mathcal{N}=\mathcal{C}_0/(\mathcal{N}\cap\mathcal{C}_0)$. In order not to clutter the notation we continue to use the same symbols for the algebra of observables and the spaces $\mathcal{N}$ etc., however one should keep in mind that the spaces $\mathcal{C}_+$ etc. for a finite and for the infinite chain cannot be identified. In particular equation (5.4) loses its meaning in the limit: the left hand side goes to zero pointwise, even though its norm in $\mathcal{H}_{OS}$ in general does not. For observables in $\mathcal{C}_+\cap\mathcal{C}_{\mathbb{T}\,\mathrm{ainv}}$ the explicit formula (4.13) can be used to compute the inner products $(\;,\;)_{OS}$. Outside this class in general the original definition $(\;,\;)_{OS}=\lim_{L\to\infty}(\;,\;)_L$ has to be used. On the other hand (5.2) always has a sensible limit:

Proposition 5.1.

(i) The limit $\lim_{L\to\infty}\mathcal{O}_{0,L}(n_0) =: \mathcal{O}_{0,\infty}(n_0)$ *exists and obeys*

$$\mathcal{O}_{0,\infty}(n_0) := \int d\Omega(n)\,(\mathbb{T}^{x_1}K^{\mathcal{O}})(n_0,n)\frac{\mathcal{P}_{-1/2}(n^\uparrow\cdot n)}{\mathcal{P}_{-1/2}(n^\uparrow\cdot n_0)}\,\lambda_\beta(0)^{-x_\ell}\,. \tag{5.14}$$

(ii) $1\!\!1_{0,\infty} = 1\!\!1$ *and* $|\mathcal{O}_{0,\infty}(n_0)| \leq \|\mathcal{O}\|$, *where* $\|\cdot\|$ *denotes the sup norm.*

(iii) If Conjecture 2.5 holds,

$$\mathcal{O} - \mathcal{O}_{0,\infty} \in \mathcal{N}\,, \tag{5.15}$$

with respect to $(\ ,\ )_{OS}$.

Proof. (i) and (ii) are straightforward. (iii), while very plausible, requires nevertheless a proof; the one given here relies on the validity of Conjecture 2.5. It suffices to show that for any $\mathcal{A} \in \mathcal{C}_-$

$$\lim_{L\to\infty} \omega_L(\mathcal{A}(\mathcal{O} - \mathcal{O}_{0,\infty})) = 0\,. \tag{5.16}$$

In order to show this, we write

$$\begin{aligned}\omega_L(\mathcal{A}(\mathcal{O} - \mathcal{O}_{0,\infty})) \;&=\\ \omega_L(\mathcal{A}(\mathcal{O} - \mathcal{O}_{0,L})) + \omega_L(\mathcal{A}(\mathcal{O}_{0,L} &- \mathcal{O}_{0,\infty})) + (\omega_\infty - \omega_L)(\mathcal{A}(\mathcal{O} - \mathcal{O}_{0,\infty}))\,.\end{aligned} \tag{5.17}$$

The first term vanishes by construction of $\mathcal{O}_{0,L}$, the third term goes to zero as $L \to \infty$ by definition of ω_∞, whereas the second term requires a closer look. In view of

$$\Big|\omega_L(\mathcal{A}(\mathcal{O}_{0,L} - \mathcal{O}_{0,\infty}))\Big| \leq \|\mathcal{A}\| \int d\Omega(n_0) \Big|\mathcal{O}_{0,\infty}(n_0) - \mathcal{O}_{0,L}(n_0)\Big| \frac{\mathcal{T}_\beta(n_0 \cdot n^\uparrow; L)^2}{\mathcal{T}_\beta(1; 2L)}\,, \tag{5.18}$$

the difference $|\mathcal{O}_{0,\infty}(n_0) - \mathcal{O}_{0,L}(n_0)|$ needs to be examined. This however has been done in section 4.2, and the proof that the right hand side of (4.19) vanishes for $L \to \infty$ carries over. This completes the proof of (iii). ■

The relation (5.15) has several important consequences, which we discuss consecutively.

For bounded observables in $\mathcal{C}_{\mathbb{T}\,\mathrm{ainv}}$ a crucial consistency condition arises from (5.15) and (4.13). Since an explicit formula for the state $\omega_\infty = \langle\ \rangle_\infty$ is known for these observables it must come out that

$$\langle \mathcal{A}\mathcal{O}\rangle_\infty = \langle \mathcal{A}\mathcal{O}_{0,\infty}\rangle_\infty \qquad \forall \mathcal{A} \in \mathcal{C}_-\,,\quad \mathcal{C}_+ \in \mathcal{O}\,, \tag{5.19}$$

using directly the limiting formulae (4.13) and (5.14). This is indeed the case: a computation shows that both sides of (5.19) reduce to

$$\begin{aligned}&\lambda_\beta(0)^{x_<^{\mathcal{A}} - x_>^{\mathcal{O}}} \int d\Omega(n) d\Omega(n') K^{\mathcal{A}}(n^\uparrow, n) \mathcal{T}_\beta(n \cdot n'; x_>^{\mathcal{A}} - x_<^{\mathcal{O}})\\ &\times \int d\Omega(n'') K^{\mathcal{O}}(n' \cdot n'') \mathcal{P}_{-1/2}(n^\uparrow \cdot n'')\,.\end{aligned} \tag{5.20}$$

Here we wrote $x_<^{\mathcal{O}}$ for the leftmost and $x_>^{\mathcal{O}}$ for the rightmost site where an observable $\mathcal{O} \in \mathcal{C}$ is supported. The consistency condition for $\mathcal{O} \in \mathcal{C}_{\mathrm{T\,ainv}}$ is therefore satisfied. On

the other hand (5.15) is valid for *all* bounded observables and the computation leading to (5.20) does not seem to leave much room for expressions other than (4.13) having the same property (5.19). This suggests that (4.13) is actually valid for all bounded observables though our proof is not.

Next let us consider asymptotically invariant observables. For them the result (5.22) yields an alternative derivation of (4.35). Since it is based on (5.22) this derivation highlights that the origin of the result (4.35) lies in the concentration property of the measures described in section 4.3. To this end we write $K^{\mathcal{O}}$ as $K^{\mathcal{O}_\infty}+(K^{\mathcal{O}}-K^{\mathcal{O}_\infty})$ and insert into the definition of $\mathcal{O}_{0,\infty}(n_0)$. Since (5.23) is trivially satisfied for the integral operators coming from $\mathrm{SO}(1,2)$ invariant operators the first term gives (4.13) with $K^{\mathcal{O}}$ replaced by $K^{\mathcal{O}_\infty}$, which is the asserted result. Using (4.34) the modulus of the second term can be bounded by

$$\lambda_\beta(0)^{-x_\ell}\frac{Q(\xi_0,x_1)}{\mathcal{P}_{-1/2}(\xi_0)}\,, \tag{5.21}$$

where $\xi_0 = n^\uparrow\cdot n_0$ and $Q(\xi_0;x_1)$ is defined in after Eq. (4.35). According to (5.22) we have to analyze the limit $n_0\to\infty$ of this expression. To this end we split the region of integration in Q into a bounded part $\xi'\in[1,\Lambda]$ and a remainder $\xi'\in[\Lambda,\infty[$. For the unbounded part we use $\overline{\mathcal{T}_\beta}(\xi_0,\xi';x_1)\le\mathcal{P}_{-1/2}(\xi_0)\mathcal{P}_{-1/2}(\xi')\,\mathcal{T}_\beta(1;x_1)$ to get a n_0 independent bound p_1/Λ on it. In the bounded part we use the fact that $\overline{\mathcal{T}_\beta}(\xi_0,\xi';x_1)$ vanishes faster than any power in ξ_0, and does so uniformly for all $\xi'\in[1,\Lambda]$. For large enough ξ_0 the supremum $\sup[\overline{\mathcal{T}_\beta}(\xi_0,\xi';x_1)/\mathcal{P}_{-1/2}(\xi_0)]$ over $\xi'\in[1,\Lambda]$ can be therefore be made smaller than $1/\Lambda^2$. The upshot is that (5.21) can be made smaller than any prescribed quantity. This completes the derivation of (4.35) based on (5.22).

A simple consequence of (5.15) is that in contrast to the finite volume case $\mathcal{C}_0$ now also intersects the null space $\mathcal{N}$: for instance all functions going to zero for $n\to\infty$ (i.e. $n^\uparrow\cdot n\to\infty$) will be mapped into the null vector of $\mathcal{H}_{OS}$, according to the previous section. The same is true for all functions that go to zero for $n\to\infty$ after averaging over $\mathrm{SO}^\uparrow(2)$. In fact, according to the discussion in Section 4.2, this exhausts the intersection $\mathcal{N}\cap\mathcal{C}_0$. Likewise products of the form $c(n)\psi(n)\in\mathcal{C}_0$, where $c(n)\to c$ for $n\to\infty$, differ from $c\psi(n)$ only by an element of $\mathcal{N}$, since their difference goes to zero as $n\to\infty$. This means that in linear combinations of vectors constant coefficients can always be replaced with coefficients satisfying this decay condition without changing the equivalence class mod $\mathcal{N}$.

Setting $\mathcal{A}=\mathbb{1}$ in (5.17) and using the results of section 4.3 one infers

$$\omega_\infty(\mathcal{O})=\omega_\infty(\mathcal{O}_{0,\infty})=w-\lim_{A\to\infty}\overline{\mathcal{O}}_{0,\infty}(An^\uparrow)\quad\text{for all}\quad\mathcal{O}\in\mathcal{C}_b\,. \tag{5.22}$$

The weak limit arises because for a 1-point observable only the behavior at infinity, defined through some unbounded sequence of A's, is relevant. This limit does not necessarily exist, however as the $\overline{\mathcal{O}}_{0,\infty}(An^\uparrow)$ form a bounded sequence in $\mathbb{R}$ one can always select a convergent subsequence. As shown in section 4.2 the limit does exist for all

$\mathcal{O} \in \mathcal{C}_{\mathbb{T}\,\mathrm{ainv}}$ without taking subsequences and is given by (4.13). For the following discussion it is convenient to introduce a somewhat smaller class of observables which we call P-invariant:

$$\mathcal{O} \in \mathcal{C}_{\mathrm{P\,inv}} \quad \text{iff} \quad PK^{\mathcal{O}} = K^{\mathcal{O}} P \,. \tag{5.23}$$

One has

$$\mathcal{C}_{\mathbb{T}\,\mathrm{inv}} \subset \mathcal{C}_{\mathrm{P\,inv}} \subset \mathcal{C}_{\mathbb{T}\,\mathrm{ainv}} \,. \tag{5.24}$$

The second inclusion is trivial; the first inclusion follows by taking the $x \to \infty$ limit $\mathcal{T}_\beta(1;x)^{-1}[K^{\mathcal{O}}, \mathbb{T}^x] = 0$. The condition (5.23) is chosen such that $\mathcal{O}_{0,\infty}(n_0)$ is independent of n_0, so that by (4.32) the value of $\mathcal{O}_{0,\infty}$ directly coincides with the thermodynamic limit of $\mathcal{O} \in \mathcal{C}_P$. By a computation similar to the one in (4.32) one shows from (5.22) that for separately $\mathrm{SO}^{\uparrow}(2)$ invariant $\overline{\mathcal{A}}, \overline{\mathcal{B}} \in \mathcal{C}^{\uparrow}_{\mathrm{P\,inv}}$ one has the 'hyperclustering' relation

$$\omega_\infty(\overline{\mathcal{A}}\,\overline{\mathcal{B}}) = \omega_\infty(\overline{\mathcal{A}})\,\omega_\infty(\overline{\mathcal{B}}) \,. \tag{5.25}$$

Since in general $\overline{\mathcal{A}\mathcal{B}} \neq \overline{\mathcal{A}}\,\overline{\mathcal{B}}$ this does not extend to all of $\mathcal{C}_{\mathrm{P\,inv}}$.

The above properties of $\mathcal{C}^{\uparrow}_{\mathrm{P\,inv}}$ observables render them at the same time uninteresting from the viewpoint of the OS reconstruction. More generally we have

Proposition 5.2. *Observables $\mathcal{O} \in \mathcal{C}_+$ are mapped onto multiples of the 'canonical' ground state ψ_0 in $\mathcal{H}_{OS}$ if and only if the following 'hyperclustering relation' holds.*

$$\omega_\infty([\vartheta\mathcal{O}]\,\mathcal{O}) = \omega_\infty(\vartheta\mathcal{O})\,\omega_\infty(\mathcal{O}) \,. \tag{5.26}$$

Sufficient conditions for (5.26) to hold are: (i) $\overline{\mathcal{O}}_{0,\infty}(An^{\uparrow})$ in (5.22) has a unique (and hence invariant) limit as $A \to \infty$. (ii) $\mathcal{O} \in \mathcal{C}^{\uparrow}_{\mathrm{ainv}} \cup \mathcal{C}^{\uparrow}_{P\mathrm{inv}}$.

Proof. The relation (5.26) is equivalent to $\mathcal{O} - \omega_\infty(\mathcal{O}) \in \mathcal{N}$ being a null vector. This in turn is equivalent to $\mathcal{O}$ and $\omega_\infty(\mathcal{O})$ giving rise to the same vector in $\mathcal{H}_{OS}$. But the latter is a multiple of the ground state, as asserted. The condition (i) is sufficient because the Cauchy-Schwarz inequality then implies $\omega_\infty([\mathcal{O} - \omega_\infty(\mathcal{O})]\mathcal{A}) = 0$ for all $\mathcal{A} \in \mathcal{C}_+$, which for $\mathcal{A} = \vartheta[\mathcal{O} - \omega_\infty(\mathcal{O})]$ amounts to (5.26). The fact that observables in $\mathcal{C}^{\uparrow}_{\mathrm{ainv}}$ or in $\mathcal{C}^{\uparrow}_{P\mathrm{inv}}$ have hyperclustering expectations has been seen before. ∎

5.3 The action of $\mathrm{SO}(1,2)$ on $\mathcal{H}_{OS}$.

Next let us consider the action of $\mathrm{SO}(1,2)$ on the reconstructed Hilbert space. Both (5.10) and (5.11) have well defined limits for $L \to \infty$ given by

$$(\rho_\infty(A)\psi)(n) \;=\; \psi(A^{-1}n)\frac{\mathcal{P}_{-1/2}(A^{-1}n\cdot n^{\uparrow})}{\mathcal{P}_{-1/2}(n\cdot n^{\uparrow})}\,, \quad \psi \in \mathcal{C}_0 \,, \tag{5.27}$$

$$(\rho_\infty(A)\mathcal{O})(n_{x_0},\ldots,n_{x_{\ell-1}}) \;=\; \mathcal{O}(A^{-1}n_{x_0},\ldots,A^{-1}n_{x_{\ell-1}})\frac{\mathcal{P}_{-1/2}(A^{-1}n_{x_{\ell-1}}\cdot n^{\uparrow})}{\mathcal{P}_{-1/2}(n_{x_{\ell-1}}\cdot n^{\uparrow})}\,, \quad \mathcal{O} \in \mathcal{C}_+ \,.$$

Here $\rho_\infty(A)$ is a well defined bounded linear map from $\mathcal{C}_+$ onto itself because the quotient $\mathcal{P}_{-1/2}(A^{-1}n\cdot n^\uparrow)/\mathcal{P}_{-1/2}(n\cdot n^\uparrow)$ is a bounded continuous function with a bounded inverse. One readily verifies the representation property $\rho_\infty(A)(\rho_\infty(B)\psi)(n) = (\rho_\infty(AB)\psi)(n)$. Further the action on $\mathcal{C}_+$ is again compatible with that on $\mathcal{C}_0$ and induces it via (5.14), namely: $(\rho_\infty(A)\mathcal{O})_0(n) = (\rho_\infty(A)\mathcal{O}_0)(n)$, for all $A \in \mathrm{SO}(1,2)$. In particular this ensures that the null space $\mathcal{N}$ and its completion are mapped onto itself under ρ_∞. For clarity's sake let us add the reminder that for $\mathcal{O} \in \mathcal{C}_+$ the assignment of $x_{\ell-1} \geq 0$ as the index of the last argument on which $\mathcal{O}$ actually depends is ambiguous since one may always consider a constant dependence on further arguments; see the comment after Eq. (3.4).

Proposition 5.3. *(i) The representation ρ_∞ of $\mathrm{SO}(1,2)$ on $\mathcal{H}_{OS}$ is uniformly bounded and measurable. (ii) It does not act unitarily on all of $\mathcal{H}_{OS}$.*

Remark 1. Uniform boundedness means that $\sup_A \|\rho_\infty(A)\psi\|_{OS} < \infty$, measurability of the representation means that the functions $A \mapsto (\psi_1, \rho_\infty(A)\psi_2)_{OS}$ and $A \mapsto (\rho_\infty(A)\psi_1, \psi_2)_{OS}$ are measurable wrt the Haar measure on $\mathrm{SO}(1,2)$.

Remark 2. The fact (ii) may be surprising at first sight, upon second thought it is not: the inner product $(\ ,\ )_{OS}$ is constructed in terms of the limiting expectation functional $w-\lim_{L\to\infty}\omega_L = \omega_\infty$, and we already know that this functional is not ρ invariant for all $\mathcal{O} \in \mathcal{C}_b$. Of course ρ_∞ is different from ρ but it seems 'unlikely' that the universal ratio $\mathcal{P}_{-1/2}(A^{-1}n\cdot n^\uparrow)/\mathcal{P}_{-1/2}(n\cdot n^\uparrow)$ by which they differ could 'undo' the symmetry breaking for *all* of the relevant observables at the same time. As a consequence the 'square root' of a bounded observable signaling the ρ symmetry breaking is likely to give rise to a wave function in $\mathcal{C}_0$ on which the unitarity of ρ_∞ is violated.

Proof of Proposition 5.3: (i) By (C.1) in fact $\|\rho_\infty(A)\psi\|_{OS} \leq \||\psi|^2\|_\infty$ using (C.1). Measurability follows from the fact that for each L the functions $A \mapsto \omega_L(\psi_1^*\,\rho_\infty(A)\psi_2)$ and $A \mapsto \omega_L((\rho_\infty(A)\psi_1^*)\,\psi_2)$ are in $L^\infty(\mathrm{SO}(1,2))$. By construction of the state $\omega_\infty(\,.\,) = w-\lim_{L\to\infty}\omega_L(\,.\,)$ the $L\to\infty$ limit of the above functions exists pointwise for almost all $A \in \mathrm{SO}(1,2)$ wrt the Haar measure. On general grounds the limiting functions are therefore measurable. As a warning we should add that for generic ψ_1, ψ_2 continuity in A may be lost in the limit, as we shall see later.

(ii) It suffices to give examples. One class is provided by wave functions only depending on the $\mathrm{SO}^\uparrow(2)$ phases. Consider $\psi_l(n) := e^{il\varphi}$, with $l \in \mathbb{Z}$ and $\varphi = \arctan(n^1/n^2)$. Then

$$0 = (\psi_l, \psi_{l'})_{OS} \neq (\rho_\infty(A)\psi_l, \rho_\infty(A)\psi_{l'})_{OS}\,, \quad l \neq l'\,. \tag{5.28}$$

An example for the square root construction mentioned in remark 2 is

$$S_e(n) := [T_e(n)]^{1/2}\,, \tag{5.29}$$

where $T_e(n)$ is the symmetry breaking observable of Eq. (4.51) and the principal branch

of the square root is taken. Then

$$\overline{T}_q(\infty) = (S_e, S_e)_{OS} \neq (\rho_\infty(A)S_e, \rho_\infty(A)S_e)_{OS} = \lim_{n^\uparrow \cdot n \to \infty} \overline{T_e(A^{-1}n)\left(\frac{\mathcal{P}_{-1/2}(n\cdot An^\uparrow)}{\mathcal{P}_{-1/2}(n\cdot n^\uparrow)}\right)^2}\,, \tag{5.30}$$

for $A \in \mathrm{SO}(1,2)/\mathrm{SO}^\uparrow(2)$ and with the overbar referring to the $\mathrm{SO}^\uparrow(2)$ average. ■

Of course one could also extend the action of ρ from $L^2(\mathbb{H})$ to $\mathcal{C}_0$ and thereby to $\mathcal{H}_{OS}$ in the obvious way. It acts, however, uninterestingly: first of all ψ_0 is mapped onto itself, likewise all elements of $\mathcal{C}^\uparrow_{\mathrm{ainv}}$ are mapped onto a multiple of ψ_0. Thus ρ acts 'unitarily' on multiples of ψ_0 by not acting at all, and since outside of the class $\mathcal{C}^\uparrow_{\mathrm{ainv}}$ symmetry breaking is generic, ρ cannot be expected to act unitarily on sizeable subspaces of $\mathcal{H}_{OS}$.

On which subspaces of $\mathcal{H}_{OS}$ does ρ_∞ act unitarily? Let us introduce the following subsets of $\mathcal{H}_{OS}$: first let $\mathcal{H}^0_{OS}$ be the closed linear subspace generated by the 'ground state orbit'

$$\{\psi \in \mathcal{H}_{OS} \,|\, \psi = \rho_\infty(A)\psi_0,\ A \in \mathrm{SO}(2,1)\}\,, \tag{5.31}$$

and $\mathcal{H}^\alpha_{OS}$ be the closed linear subspace generated by

$$\{\psi \in \mathcal{H}_{OS} \,|\, \psi = \rho_\infty(A)\psi_\alpha,\ A \in \mathrm{SO}(2,1)\}\,, \quad \alpha \in \mathbb{R}\setminus\{0\}\,, \tag{5.32}$$

with $\psi_\alpha(n) = \exp(i\alpha\, n^\uparrow\!\cdot n)$. $\mathcal{H}^0_{OS}$ does not change if we allow the coefficients to be from $\mathcal{C}^\uparrow_{\mathrm{ainv}}$. $\mathcal{H}^0_{OS}$ and $\mathcal{H}^\alpha_{OS}$ are by construction invariant subspaces of $\mathcal{H}_{OS}$ under the action of the representation ρ_∞. It is convenient to introduce the notation

$$\psi_{n_0,\alpha}(n) := \frac{\mathcal{P}_{-1/2}(n_0\cdot n)}{\mathcal{P}_{-1/2}(n^\uparrow \cdot n)}\, e^{i\alpha n_0 \cdot n}\,, \quad n_0 \in \mathbb{H}\,,\ \ \alpha \neq 0\,, \tag{5.33}$$

for the basis vectors; then ρ_∞ acts simply by 'rotating' n_0, i.e. $\psi_{n_0,\alpha} \to \psi_{An_0,\alpha}$, $A \in \mathrm{SO}(1,2)$. Note that this action inherits the properties of the action of $\mathrm{SO}(1,2)$ on $\mathbb{H}$. As such it is transitive and effective but not free. It is not free because $An = n$ for some fixed $n \in \mathbb{H}$ implies only that A is in stability group of n. The action is manifestly transitive and also effective in that $An = n$ for all $n \in \mathbb{H}$ implies $A = \mathbb{1}$. We define $\mathcal{H}^{\mathrm{p}}_{OS}$ (p being mnemonic for 'phase' or 'polymer') as the closed subspace generated by all the vectors $\psi_{n_0,\alpha}$, $\alpha \neq 0$, in (5.33).

We now describe how ρ_∞ acts on $\mathcal{H}^0_{OS}$ and $\mathcal{H}^{\mathrm{p}}_{OS}$:

Theorem 5.4. *$\mathcal{H}^0_{OS}$ and $\mathcal{H}^{\mathrm{p}}_{OS}$ are orthogonal subspaces of $\mathcal{H}_{OS}$. ρ_∞ acts unitarily on both of these subspaces; the action is continuous on $\mathcal{H}^0_{OS}$, but discontinuous on $\mathcal{H}^{\mathrm{p}}_{OS}$. Furthermore, on $\mathcal{H}^0_{OS}$ one has*

$$(\rho_\infty(A)\psi_0, \rho_\infty(B)\psi_0)_{OS} = \mathcal{P}_{-1/2}(An^\uparrow \cdot Bn^\uparrow)\,, \tag{5.34}$$

whereas on $\mathcal{H}^{\rm p}_{OS}$ one has

$$(\psi_{n_1,\alpha_1}\,,\,\psi_{n_2,\alpha_2})_{OS} = \delta_{n_1,n_2}\delta_{\alpha_1,\alpha_2}\,, \quad \forall\, n_1,n_2\in\mathbb{H}\,,\;\; \alpha_1,\alpha_2\in\mathbb{R}\,,\; \alpha_1\alpha_2\neq 0\,. \tag{5.35}$$

Proof. The derivation of Eqs (5.34), (5.35) as well as the proof of the orthogonality of the two subspaces is somewhat technical and is deferred to appendix C.

From (5.34) it is easy to see that ρ_∞ acts continuously on $\mathcal{H}^0_{OS}$: By (5.34) the scalar product of any two elements in the orbit of ψ_0 is a continuous function of the group elements, and this continuity trivially extends to finite linear combinations of elements of this orbit. Denote this linear space by $\mathcal{D}$. This implies that for any $\phi\in\mathcal{D}$ we have $\lim_{A\to 0}\|\rho_\infty(A)\phi-\phi\|_{OS}=0$. Now for any element $\psi\in\mathcal{H}^0_{OS}$ and any $\epsilon>0$ there is a $\phi_\epsilon\in\mathcal{D}$ such that $\|\phi_\epsilon-\psi\|_{OS}<\epsilon$. By the triangle inequality therefore $\lim_{A\to 0}\|\rho_\infty(A)\psi-\psi\|_{OS}\leq\epsilon$, and since ϵ was arbitrary, $\lim_{A\to 0}\|\rho_\infty(A)\psi-\psi\|_{OS}=0$ follows. The group structure of SO(1,2) yields (strong) continuity of the whole representation $\rho_\infty\big|_{\mathcal{H}^0_{OS}}$. The discontinuity of the action of ρ_∞ on $\mathcal{H}^{\rm p}_{OS}$ is obvious from (5.35). ∎

Corollary 5.5 *$\mathcal{H}_{OS}$ is nonseparable.*

Proof. The vectors (5.33) provide an explicit nondenumerable orthonormal family. ∎

Remark. The representation ρ_∞ of $\mathrm{SO}(1,2)$ acts as a kind of 'nondenumerable discrete permutation group', $\rho_\infty(A)\psi_{n_0,\alpha}=\psi_{An_0,\alpha}$, on the orthonormal family (5.33) (see Appendix C).

The above result should be viewed in the context of an alternative described by Segal and Kunze ([46], p. 274) which characterizes measurable unitary representations π of some locally compact group G on a nonseparable Hilbert space $\mathcal{H}$. Namely let $\mathcal{H}^s$ be the subspace of all vectors ψ_s in $\mathcal{H}$ such that for all $\varphi\in L^1(G)$ and all $\psi\in\mathcal{H}$ one has

$$\int_G d\mu(A)\,\varphi(A)(\psi,\pi(A)\psi_s)=0\,. \tag{5.36}$$

Then $\mathcal{H}$ is the direct sum of two invariant subspaces $\mathcal{H}=\mathcal{H}^c\oplus\mathcal{H}^s$. The restriction of π to $\mathcal{H}^c$ is continuous while the restriction to $\mathcal{H}^s$ is singular, in the sense that $(\psi_s,\pi(A)\psi_s)=0$ for almost all $A\in G$ and all $\psi_s\in\mathcal{H}^s$. If $\mathcal{H}$ is separable $\mathcal{H}^s$ is absent, as follows from a theorem of von Neumann (see [30], Theorem VIII.9).

In our case we denote by $\mathcal{H}^u_{OS}$ the maximal closed subspace of $\mathcal{H}_{OS}$ on which ρ_∞ is unitary and measurable. The above alternative entails that $\mathcal{H}^u_{OS}$ decomposes into $\mathcal{H}^u_{OS}=\mathcal{H}^c_{OS}\oplus\mathcal{H}^s_{OS}$, where the restriction of ρ_∞ to $\mathcal{H}^c_{OS}$ and $\mathcal{H}^s_{OS}$ is continuous and singular, respectively. Our results amount to the explicit construction of subspaces

$$\mathcal{H}^0_{OS}\subset\mathcal{H}^c_{OS}\,,\qquad \mathcal{H}^{\rm p}_{OS}\subset\mathcal{H}^s_{OS}\,, \tag{5.37}$$

together with a formula for the inner products. In particular the singular subspace is non-empty (for which the nonseparability of the Hilbert space is a necessary but not a sufficient condition). The assumption (5.36) is satisfied for $\psi \in \mathcal{H}^{\rm p}_{OS} \subset \mathcal{H}^{s}_{OS}$ because (5.35) projects onto a 1-dimensional submanifold of the group which has zero measure wrt the full Haar measure. The restriction of ρ_∞ to $\mathcal{H}^{\rm p} \subset \mathcal{H}^{s}$ is indeed singular; in fact by (5.35) $(\psi_{n_0,\alpha}, \rho_\infty(A)\psi_{n_0,\alpha})_{OS} = 0$ holds for all $A \neq 1\!\!1$. The simple explicit action $\rho_\infty(A)\psi_{n_0,\alpha} = \psi_{An_0,\alpha}$ as a permutation group (acting transitively and effectively for fixed α) is somewhat surprising. The continuous subspace $\mathcal{H}^0_{OS}$ will later be identified as the ground state sector of the reconstructed transfer operator.

As outlined in appendix C there are other nondenumerable orthonormal families in $\mathcal{H}_{OS}$ which are orthogonal to both $\mathcal{H}^0_{OS}$ and $\mathcal{H}^{\rm p}_{OS}$. We did not explore the action of ρ_∞ on them, but it may well be that $\mathcal{H}_{OS}$ contains other invariant subspaces on which ρ_∞ acts unitarily. In this case they would likewise be subject to the above continuous-discontinuous alternative and render the inclusions in (5.37) proper.

We proceed with the construction of a transfer operator $\mathbb{T}_{OS}$ on $\mathcal{H}_{OS}$, which in particular will justify the term 'ground state sector' for $\mathcal{H}^0_{OS}$ (see Proposition 5.6 below). To this end let τ be the map from $\mathcal{C}_+$ to $\mathcal{C}_+$ that shifts all variables by 1 unit to the right, i.e. $\tau\mathcal{O}(n_{x_1}, \ldots, n_{x_\ell}) = \mathcal{O}(n_{x_1+1}, \ldots n_{x_\ell+1})$. τ satisfies the relation $\tau\vartheta\tau = \vartheta$; it maps $\mathcal{N}$ into itself as can be seen by using the Cauchy-Schwarz inequality and translation invariance

$$\omega_\infty(\vartheta[\tau\mathcal{O}]\tau\mathcal{O}) = \omega_\infty([\vartheta\mathcal{O}]\,[\tau^2\mathcal{O}]) \leq \omega_\infty([\vartheta\mathcal{O}]\ [\tau^4\mathcal{O}])^{1/2}\,\omega_\infty([\vartheta\mathcal{O}]\,O)^{1/2}\,. \tag{5.38}$$

τ therefore induces a well-defined operator $\mathbb{T}_{OS}$ on the equivalence classes modulo $\mathcal{N}$, and hence on $\mathcal{H}_{OS}$. By translation invariance $\mathbb{T}_{OS}$ is symmetric. Once known to be bounded it extends to a unique selfadjoint operator on $\mathcal{H}_{OS}$. The boundedness follows by iterating the Cauchy-Schwarz inequality (using a classic argument of Osterwalder and Schrader)

$$\begin{aligned}\omega_\infty([\vartheta\mathcal{O}]\,\tau^2\mathcal{O}) \ &\leq\ \omega_\infty([\vartheta\mathcal{O}]\,\tau^4\mathcal{O})^{1/2}\omega_\infty([\vartheta\mathcal{O}]\,\mathcal{O})^{1/2} \leq \ldots \\ &\leq\ \omega_\infty([\vartheta\mathcal{O}]\,\tau^{2^{n+1}}\mathcal{O})^{2^{-n}}\omega_\infty([\vartheta\mathcal{O}]\,\mathcal{O})^{1-2^{-n}}\,. \end{aligned}\tag{5.39}$$

The first factor is bounded by $\|\mathcal{O}\|^{2^{-n+1}}$, which goes to 1 as $n \to \infty$; the second factor goes to $\omega_\infty([\vartheta\mathcal{O}]\,\mathcal{O})$, which proves that $\|\mathbb{T}^2_{OS}\| \leq 1$ and thus also $\|\mathbb{T}_{OS}\| \leq 1$.

Importantly, the vector ψ_0 corresponding to $\mathcal{O} = 1\!\!1$ is an eigenvector (of norm 1) of the reconstructed transfer operator $\mathbb{T}_{OS}$ with eigenvalue $1 = \|\mathbb{T}_{OS}\|_{OS}$, i.e. ψ_0 is a ground state of the system. Already the mere existence of at least one normalizable ground state indicates that the reconstructed quantum mechanics given by $(\mathcal{H}_{OS}, \mathbb{T}_{OS})$ is very different from the original one given by $(L^2(\mathbb{H}), \mathbb{T})$. This mismatch can be traced back to the purely continuous spectrum of the original system, which in turn stems from the noncompactness of the target space $\mathbb{H}$. A further drastic discrepancy is the nonseparability of $\mathcal{H}_{OS}$.

Similar surprising features arise already in the much simpler model with flat target space $\mathbb{R}$, on which $\mathbb{R}$ also acts as an amenable symmetry. This example is also instructive because it shows that in the limit of an amenable symmetry the symmetry breaking disappears. We therefore discuss this example briefly in Appendix B.

Returning to the hyperbolic model, we summarise the properties of $\mathbb{T}_{OS}$:

Proposition 5.6. *$\mathbb{T}_{OS}$ is a self-adjoint operator on $\mathcal{H}_{OS}$ with following properties:*

(i) $||\mathbb{T}_{OS}|| = 1$.

(ii) $\rho_\infty \circ \mathbb{T}_{OS} = \mathbb{T}_{OS} \circ \rho_\infty$.

(iii) $\mathbb{T}_{OS}\big|_{\mathcal{H}^0_{OS}} = \mathbb{1}\big|_{\mathcal{H}^0_{OS}}$.

(iv) $\mathcal{H}^0_{OS} = \{\psi \in \mathcal{H}_{OS} \,|\, \mathbb{T}_{OS}\psi = \psi\}$.

(v) $\mathbb{T}_{OS}$ *acts on* $\mathcal{C}_+/\mathcal{N}$*, i.e. on the representatives (5.14) as*

$$(\mathbb{T}^x_{OS}\psi)(n) = \lambda_\beta(0)^{-x}\mathcal{P}_{-1/2}(n\cdot n^\uparrow)^{-1}\int d\Omega(n')\mathcal{T}_\beta(n\cdot n';x)\psi(n')\mathcal{P}_{-1/2}(n'\cdot n^\uparrow)\,, \quad (5.40)$$

up to an element of $\mathcal{N}$*.*

Remark. (ii) and (iv) show that $\mathbb{T}_{OS}$, in contrast with $\mathbb{T}$, has at least some point spectrum. Despite the concrete expression in (v), it seems difficult to say more about the spectrum of $\mathbb{T}_{OS}$ outside the vacuum space $\mathcal{H}^0_{OS}$.

Proof. (i) has already been shown; it is a general feature of the Osterwalder-Schrader reconstruction.

(ii) Recall that $\mathbb{T}_{OS}$ is defined in terms of the shift τ on $\mathcal{C}_+$. As τ trivially commutes with the ρ_∞ action (5.27) of $\mathrm{SO}(1,2)$ on $\mathcal{C}_+$ and both τ and ρ_∞ preserve the nullspace $\mathcal{N}$, the same will be true for $\mathbb{T}_{OS}$ induced on the equivalence classes. This gives (ii).

(iii) A simple consequence of (ii) is that $\mathbb{T}_{OS}$ acts like the identity on $\mathcal{H}^0_{OS}$, because τ acts like the identity on the constants, in particular on the unit observable $\mathcal{O} = \mathbb{1}$, corresponding to the 'canonical' vacuum ψ_0. By (ii) the same must hold for all elements of $\mathcal{H}^0_{OS}$. Equivalently, the eigenspace of $\mathbb{T}_{OS}$ of eigenvalue 1 contains $\mathcal{H}^0_{OS}$.

(iv) Let $\mathcal{O} \in \mathcal{C}_+$ be such $\tau\mathcal{O} - \mathcal{O} \in \mathcal{N}$. Then also $\tau\mathcal{O}_0 - \mathcal{O}_0 \in \mathcal{N}$, because $\tau\mathcal{O} - \mathcal{O} = \tau\mathcal{O}_0 - \mathcal{O}_0 + \tau(\mathcal{O} - \mathcal{O}_0) - (\mathcal{O} - \mathcal{O}_0)$, and the last two terms on the rhs are in $\mathcal{N}$. By the remark after (5.14) therefore $(\tau\mathcal{O}_0)_0 - \mathcal{O}_0 \in \mathcal{N} \cap \mathcal{C}_0$, and it suffices to consider the exact identity $(\tau\mathcal{O}_0)_0 = \mathcal{O}_0$. From the definition of the map (5.14) one sees that all solutions $\psi \in \mathcal{C}_0$ of $(\tau\psi)_0 = \psi$ are such that $\psi(n)\mathcal{P}_{-1/2}(n \cdot n^\uparrow)$ is an eigenfunction of $\mathcal{T}_\beta$ of eigenvalue $\lambda_\beta(0)$. From (2.42) one infers that the solutions lie in the closed subspace of $\mathcal{C}_0$ spanned by ratios of the form (5.33) with $\alpha = 0$.

(v) Since $(\tau\mathcal{O}_0)_0 - (\tau\mathcal{O})_0 \in \mathcal{N}$, for all $\mathcal{O} \in \mathcal{C}_+$, one can use (5.14) to compute the action of τ on the representative $\mathcal{O}_0$ as before. One finds (5.40), first for $x = 1$ and then by iteration for all $x \in \mathbb{N}$. ■

In addition to acting unitarily, ρ_∞ also acts irreducibly on $\mathcal{H}^0_{OS}$. This follows directly from the definition (5.31). Alternatively one can use the addition theorem (A.12c) to replace the generating set $\rho_\infty(A)\psi_0$, $A \in \mathrm{SO}(1,2)$, by the alternative generating set $\mathcal{P}^l_{-1/2}(\xi)/\mathcal{P}_{-1/2}(\xi)$, $l \in \mathbb{Z}$. These functions are known to span an irreducible and unitary representation of $\mathrm{SO}(1,2)$; in the Bargmann classification it corresponds to the limit of the discrete series.

To sum up, we have found that the space $\mathcal{H}_{OS}$ is nonseparable and that it carries a representation ρ_∞ of the symmetry group. This representation acts unitarily and discontinuously on a nonseparable proper subspace of $\mathcal{H}_{OS}$, and unitarily and continuously on the separable subspace of ground states $\mathcal{H}^0_{OS}$ of the reconstructed transfer operator $\mathbb{T}_{OS}$. This ground state sector is irreducible and can be described explicitly as

$$\mathcal{H}^0_{OS} \simeq \left\{ \frac{\mathcal{P}_{-1/2}(n^\uparrow \cdot A^{-1} n)}{\mathcal{P}_{-1/2}(n^\uparrow \cdot n)} \middle| A \in \mathrm{SO}(1,2) \right\} \simeq \left\{ \frac{\mathcal{P}^l_{-1/2}(n^\uparrow \cdot n)}{\mathcal{P}_{-1/2}(n^\uparrow \cdot n)} \middle| l \in \mathbb{Z} \right\} \subset \mathcal{H}_{OS}\,, \tag{5.41}$$

where the symbol '$\simeq$' denotes equality of the span of the lhs and rhs for the equivalence classes modulo $\mathcal{N}$. The inner product on $\mathcal{H}^0_{OS}$ is given by $(\rho_\infty(A)\psi_0, \rho_\infty(B)\psi_0)_{OS} = \mathcal{P}_{-1/2}(An^\uparrow \cdot Bn^\uparrow)$.

6. Conclusions and outlook

We have found that the concept of spontaneous symmetry breaking for a nonamenable continuous internal symmetry group differs in some crucial ways from the familiar situation of an amenable symmetry group:

- Symmetry breaking is unavoidable, even in dimensions 1 and 2, where it is forbidden for an amenable continuous symmetry. In one dimension the (improper) ground state in the quantum mechanical interpretation is infinitely degenerate; in the statistical mechanics interpretation invariant states over a 'large' algebra cannot be defined by group averaging. These features have been worked out in some detail for an analytically solvable model, the hyperbolic spin chain with symmetry group $\mathrm{SO}(1,2)$.

- In this 1-dimensional model, however, there is still some vestige of the large fluctuations that are responsible for the symmetry restoration in the compact and abelian models: the sequence of functional measures defined through the thermodynamic limit becomes concentrated at configurations 'at infinity' of the hyperbolic plane.

As a consequence a certain subclass of non-invariant observables gets averaged to yield an invariant result. The limit of the functional measures provides an invariant mean for this subclass of observables, while outside this class symmetry breaking is generic.

- While the quantum mechanics described by our model can be simply interpreted as motion of a particle on the hyperbolic plane, with absolutely continuous spectrum of the transfer matrix, the Osterwalder-Schrader reconstruction based on the infinite volume expectation values yields some surprises: the reconstructed transfer matric has at least some point spectrum, in particular it has normalizable ground states, and the full reconstructed Hilbert space is nonseparable. These features are, however, due to the noncompact nature of the symmetry group, not its nonamenability, as can be seen from the 'flat' analogue discussed in Appendix B.

- The Osterwalder-Schrader reconstructed Hilbert space has a nonseparable proper subspace on which a unitary representation of $\mathrm{SO}(1,2)$ acts discontinuously as a kind of 'nondenumerable discrete permutation group', not unlike the way the spatial diffeomorphism group acts on the embedded graphs in the framework of [22, 23]. In contrast, the space of ground states of the reconstructed transfer operator is separable and a nontrivial unitary representation of $\mathrm{SO}(1,2)$ acts on it continuously and irreducibly. These features are specific to the case of a nonamenable symmetry and are not present in the 'flat' case.

In a follow-up paper we study these issues in the D-dimensional ($D \geq 2$) version of the model, i.e. the nonlinear sigma-model with a hyperbolic targetspace; see e.g. [18, 42, 19] for earlier investigations. There we use a combination of analytical techniques and of numerical simulations [14]. We also expect that there will still be a marked difference between dimensions $D \leq 2$ and $D \geq 3$: whereas in the low dimensional case there is, as stated above, dominance of highly boosted configurations, we expect that in $D \geq 3$ spontaneous symmetry breaking in the usual sense takes place, showing normal, approximately Gaussian fluctuations around a fully ordered state, in which for instance unbounded observables like n_0^0 have finite expectation values. Some time after the first version of this paper was posted on the web, a paper by Spencer and Zirnbauer [47] appeared, which showed that indeed in dimensions $D \geq 3$ at low temperature the suitable defined spin fluctuations have finite moments.

It would be interesting to elucidate the physical meaning of the unavoidable spontaneous symmetry breaking in the context of Anderson localization, in which such nonlinear sigma models were studied for instance in [7, 8, 9, 10, 11].

In order not to blur the discussion with (further) technicalities we contrasted here only the simplest compact and noncompact symmetric spaces. However the situation would be similar if the sphere $S^2 \simeq \mathrm{SO}(3)/\mathrm{SO}(2)$ and $\mathbb{H} \simeq \mathrm{SO}(1,2)/\mathrm{SO}(2)$ were replaced with any other dual pair of compact and noncompact riemannian symmetric spaces (see [26] for the propagators). A further generalization would be to consider a similar

dynamical system where the variables take values in an arbitrary riemannian manifold. In particular this would allow one to examine the interplay between invariant dynamics and non-invariant states for the diffeomorphism group of the target manifold.

Finally we cannot resist mentioning a potential application to quantum gravity. Supposing that in a suitable topology an appropriate version of the diffeomorphism group is nonamenable, variants of the above concepts become applicable. This would suggest a scenario in which there is no diffeomorphism invariant ground state, yet a family of selected observables has invariant expectations in each of an infinite set of ground states, while outside this family spontaneous collapse of diffeomorphism invariance is generic.

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Appendix A: Harmonic analysis on $\mathbb{H}$

Let $a{\cdot}b = a^0b^0 - a^1b^1 - a^2b^2$ be the bilinear form of $\mathbb{R}^{1,2}$ and let $\mathrm{SO}_0(1,2) =: \mathrm{SO}(1,2)$ be the component of its symmetry group connected to the identity. Consider the hyperboloid $\mathbb{H} = \{n \in \mathbb{R}^{1,2} \,|\, n{\cdot}n = 1\,, n^0 > 0\}$. It is isometric to the symmetric space $\mathrm{SO}(1,2)/\mathrm{SO}(2)$ and can be parameterized either by points (Δ, B), $\Delta > 0, B \in \mathbb{R}$, in the Poincaré upper half plane, or by geodetic polar coordinates (ξ, φ), $\xi \geq 1, -\pi \leq \varphi < \pi$, via

$$
\begin{aligned}
n^0 &= \frac{1+\Delta^2+B^2}{2\Delta} = \xi\,, \qquad n^1 = -\frac{B}{\Delta} = \sqrt{\xi^2-1}\sin\varphi\,, \\
n^2 &= \frac{-1+\Delta^2+B^2}{2\Delta} = \sqrt{\xi^2-1}\cos\varphi\,.
\end{aligned} \tag{A.1}
$$

The (ξ, φ) parameterization is adapted to a preferred $\mathrm{SO}(2)$ subgroup of $\mathrm{SO}(1,2)$ which leaves $n^\uparrow = (1,0,0)$ invariant and which we denote by $\mathrm{SO}^\uparrow(2)$. We also note the relations $\Delta^{-1} = \xi - \sqrt{\xi^2-1}\cos\varphi$, $B = \sqrt{1-\xi^{-2}}\sin\varphi/(\sqrt{1-\xi^{-2}}\cos\varphi - 1)$. For the invariant distance $n \cdot n' \geq 1$ of two points $n, n' \in \mathbb{H}$, one has

$$
n \cdot n' = \frac{\Delta^2 + \Delta'^2 + (B-B')^2}{2\Delta\Delta'} = \xi\xi' - (\xi^2-1)^{1/2}(\xi'^2-1)^{1/2}\cos(\varphi-\varphi')\,. \tag{A.2}
$$

Function spaces on $\mathbb{H}$ come naturally equipped with the inner product

$$
(\psi_1, \psi_2) = \int d\Omega(n)\, \psi_1(n)^* \psi_2(n)\,, \tag{A.3}
$$

induced by the invariant measure $d\Omega(n) := 2d^3n\,\delta(n^2-1)\,\theta(n^0)$, which translates into $d\Omega(\Delta, B) = dBd\Delta\Delta^{-2}$ and $d\Omega(\xi, \varphi) = d\xi d\varphi$, respectively. As indicated we shall freely switch back and forth between the different parameterizations. The Schwartz space $\mathcal{S}(\mathbb{H})$ is defined as the space of smooth functions on $\mathbb{H}$ decaying faster than any power of B and Δ. The space of tempered distributions $\mathcal{S}'(\mathbb{H})$ on it together with $L^2(\mathbb{H})$ form a Gel'fand space triple

$$
\mathcal{S}(\mathbb{H}) \subset L^2(\mathbb{H}) \subset \mathcal{S}'(\mathbb{H})\,. \tag{A.4}
$$

The $\mathrm{SO}(1,2)$ rotations of the 'spins' n induce a unitary representation ρ on $\mathcal{S}(\mathbb{H})$ via $\rho(A)\psi(n) = \psi(A^{-1}n)$, $A \in \mathrm{SO}(1,2)$. On integral operators K with kernel $\kappa(n, n')$ it acts as $K \to \rho(A)^{-1}K\rho(A)$ and thus as $\kappa(n,n') \to \kappa(An, An')$ on the kernels. Invariant operators have kernels depending on the inner product $n \cdot n'$ only. Similarly operators invariant under ρ restricted to the $\mathrm{SO}^\uparrow(2)$ subgroup have kernels depending on ξ, ξ' and the relative angle $\varphi - \varphi'$ only. In general the representation ρ will not be irreducible. Generic functions in $\mathcal{S}(\mathbb{H})$ can be expanded into a generalized Fourier integral whose basis functions form unitary irreps of $\mathrm{SO}(1,2)$. Moreover these basis functions comprise $\mathcal{S}'(\mathbb{H})$ eigenfunctions of the Laplace-Beltrami operator.

To make this concrete consider the Killing vectors of $\mathbb{H}$, which generate the Lie algebra sl_2

$$\begin{aligned}
&\mathbf{e} = \partial_B \,, && \mathbf{h} = 2(B\partial_B + \Delta\partial_\Delta)\,, && \mathbf{f} = (\Delta^2 - B^2)\partial_B - 2B\Delta\partial_\Delta\,, \\
&[\mathbf{h},\mathbf{e}] = -2\mathbf{e}\,, && [\mathbf{h},\mathbf{f}] = 2\mathbf{f}\,, && [\mathbf{f},\mathbf{e}] = \mathbf{h}\,,
\end{aligned} \tag{A.5}$$

and are anti-hermitian wrt $(\ ,\ )$. Up to a sign the quadratic Casimir coincides with the Laplace-Beltrami operator

$$-\mathbf{C} := \frac{1}{4}\mathbf{h}^2 + \frac{1}{2}(\mathbf{ef}+\mathbf{fe}) = \Delta^2(\partial_\Delta^2 + \partial_B^2) = \frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi} + \frac{1}{\xi^2-1}\frac{\partial^2}{\partial\varphi^2}\,. \tag{A.6}$$

If one just blindly lets the differential operators $\mathbf{e}$, $\mathbf{h}$, $\mathbf{f}$ act on the spins (A.1) (which are not elements of $L^2(\mathbb{H})$) one sees that they act as 3×3 matrices $t(\mathbf{e})$, $t(\mathbf{h})$, $t(\mathbf{f})$ with Casimir $\mathbf{C} = -2\mathbb{1}_3$; the matrices are however not (anti-)hermitian even though the original differential operators (multiplied by i) are essentially self-adjoint on $\mathcal{S}(\mathbb{H}) \subset L^2(\mathbb{H})$. The exponentiated differential operators therefore extend to the unitary action of $\mathrm{SO}(1,2)$ on $L^2(\mathbb{H})$

$$\rho(e^{-st(\mathbf{x})})\psi(n) = e^{s\mathbf{x}}\psi(n) = \psi(e^{st(\mathbf{x})}n)\,, \qquad \mathbf{x} = \mathbf{e},\,\mathbf{h},\,\mathbf{f};\quad s\in\mathbb{R}\,. \tag{A.7}$$

A more explicit description of the exponentiated differential operators is possible on irreducible representations.

Simultaneous eigenstates of $\mathbf{C}$ and $\mathbf{e}$ are given by

$$\begin{aligned}
&\epsilon_{\omega,k}(n) := \epsilon_{\omega,k}(\Delta,B) = \Delta^{1/2}K_{i\omega}(|k|\Delta)\,e^{ikB}\,, \quad k\neq 0\,. \\
&\epsilon_{\omega,0}(n) := \epsilon_{\omega,0}(\Delta,B) = \Delta^{i\omega+1/2}\,, \qquad \text{with} \\
&\mathbf{C}\,\epsilon_{\omega,k} = \Big(\frac{1}{4}+\omega^2\Big)\epsilon_{\omega,k}\,, \qquad \mathbf{e}\,\epsilon_{\omega,k} = ik\,\epsilon_{\omega,k}\,, \quad \omega>0\,,
\end{aligned} \tag{A.8}$$

where $K_\nu(x)$ is a modified Bessel function defined e.g. by $K_\nu(\beta) = \int_0^\infty e^{-\beta\ \cosh t}\cosh(\nu t)dt$. The Fourier inversion on $\mathcal{S}(\mathbb{H})$ takes the form

$$\begin{aligned}
\psi(n) &= \int_0^\infty \frac{d\omega}{\pi^3}\,\omega\sinh\pi\omega\int_{\mathbb{R}} dk\,\widehat{\psi}(\omega,k)\,\epsilon_{\omega,k}(n) \\
\widehat{\psi}(\omega,k) &= \int d\Omega(n)\,\psi(n)\epsilon_{\omega,k}(n)^*\,.
\end{aligned} \tag{A.9}$$

Simultaneous eigenstates of $\mathbf{C}$ and $\mathbf{e}-\mathbf{f}$, i.e. of the $\mathrm{SO}^\uparrow(2)$ rotations are given by

$$\begin{aligned}
&\epsilon_{\omega,l}(n) := \epsilon_{\omega,l}(\xi,\varphi) = e^{il\varphi}\,\mathcal{P}^l_{-1/2+i\omega}(\xi)\,, \qquad l\in\mathbb{Z}\,,\ \ \omega>0\,, \\
&\mathbf{C}\,\epsilon_{\omega,l} = \Big(\frac{1}{4}+\omega^2\Big)\epsilon_{\omega,l}\,, \qquad (\mathbf{e}-\mathbf{f})\,\epsilon_{\omega,l} = il\,\epsilon_{\omega,l}\,,
\end{aligned} \tag{A.10}$$

where $\mathcal{P}^l_s(\xi)$ are Legendre functions, defined e.g. by

$$\mathcal{P}^l_s(\xi) = \frac{\Gamma(s+l+1)}{2\pi\Gamma(s+1)} \int_0^{2\pi} du\, e^{ilu} [\xi + \sqrt{\xi^2-1}\cos u]^s\,, \qquad \xi \geq 1\,. \tag{A.11}$$

We further note the following properties

$$\mathcal{P}^l_s(\xi) = \mathcal{P}^l_{-s-1}(\xi) = \frac{\Gamma(s+1+l)}{\Gamma(s+1-l)} \mathcal{P}^{-l}_s(\xi)\,, \tag{A.12a}$$

$$\int_1^\infty d\xi\, \mathcal{P}^l_{-1/2+i\omega}(\xi) \mathcal{P}^{-l}_{-1/2+i\omega'}(\xi) = \frac{(-)^l}{\omega \tanh \pi\omega}\, \delta(\omega-\omega')\,, \tag{A.12b}$$

$$\mathcal{P}_s\Big(\xi\xi' - (\xi^2-1)^{1/2}(\xi'^2-1)^{1/2}\cos\varphi\Big) = \sum_{l\in\mathbb{Z}} (-)^l e^{il\varphi}\, \mathcal{P}^{-l}_s(\xi)\, \mathcal{P}^l_s(\xi')\,, \tag{A.12c}$$

as well as the asymptotics for $\xi \to \infty$

$$\begin{aligned} \mathcal{P}^l_{-1/2+i\omega}(\xi) \;&\sim\; \frac{\Gamma(i\omega)}{\sqrt{\pi}\Gamma(\frac{1}{2}+i\omega-l)} (2\xi)^{-1/2+i\omega} + c.c.\,, \quad \omega>0\,, \\ \mathcal{P}^l_{-1/2}(\xi) \;&\sim\; \frac{2}{\sqrt{\pi}\Gamma(\frac{1}{2}-l)} \frac{\ln\xi}{\sqrt{2\xi}}\,. \end{aligned} \tag{A.13}$$

The Fourier inversion in the basis (A.14) takes the form

$$\begin{aligned} \psi(n) \;&=\; \sum_{l\in\mathbb{Z}} (-)^l \int_0^\infty \frac{d\omega}{2\pi}\, \omega \tanh \pi\omega\, \widehat{\psi}(\omega,l)\, \epsilon_{\omega,-l}(n)\,, \\ \widehat{\psi}(\omega,l) \;&=\; \int d\Omega(n)\, \psi(n) \epsilon_{\omega,l}(n)\,. \end{aligned} \tag{A.14}$$

In group theoretical terms the expansions (A.9), (A.14) correspond to the decomposition of the unitary representation ρ on $L^2(\mathbb{H})$ into a direct integral of unitary irreducible representations, namely those of the type 0 principal series in the Bargmann classification, see e.g. [43]. In terms of the representation spaces

$$L^2(\mathbb{H}) = \int^\oplus d\mu(\omega)\, \mathcal{C}_\omega(\mathbb{H})\,, \tag{A.15}$$

with the spectral weight $d\mu(\omega) = \frac{d\omega}{2\pi}\, \omega \tanh \omega$. We shall frequently encounter $\mathrm{SO}^\uparrow(2)$ invariant functions $\psi = \psi(\xi)$, for which (A.14) reduces to the Mehler-Fock transform

$$\begin{aligned} \psi(\xi) \;&=\; \int_0^\infty \frac{d\omega}{2\pi}\, \omega \tanh(\pi\omega)\, \mathcal{P}_{-1/2+i\omega}(\xi)\, \widehat{\psi}(\omega)\,, \\ \widehat{\psi}(\omega) \;&=\; 2\pi \int_1^\infty d\xi\, \mathcal{P}_{-1/2+i\omega}(\xi)\, \psi(\xi) = \widehat{\psi}(\omega,0)\,. \end{aligned} \tag{A.16}$$

It holds in the classical sense provided

$$\int_1^\infty d\xi |\psi(\xi)|^2 < \infty \quad \Longleftrightarrow \quad \int_0^\infty |\widehat{\psi}(\omega)|^2 \omega \tanh \pi\omega < \infty \,, \tag{A.17}$$

see e.g. [44]. It is possible, however, to interprete the Mehler-Fock transform in the distributional sense and therefore give it a wider range of applicability.

The Fourier decomposition of a kernel $\kappa(n,n')$ defining an integral operator K makes some of its properties manifest. Subject to suitable regularity conditions the generic form of the expansion wrt the basis (A.14) is

$$\kappa(n,n') = \sum_{l_1,l_2 \in \mathbb{Z}} (-)^{l_1+l_2} \int_0^\infty \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} \, \omega_1 \mathrm{th}\pi\omega_1 \, \omega_2 \mathrm{th}\pi\omega_2 \, \widehat{\kappa}_{l_1.l_2}(\omega_1,\omega_2) \, \epsilon_{\omega_1,-l_1}(n) \, \epsilon_{\omega_2,-l_2}(n') \,, \tag{A.18}$$

Depending on the properties of the spectral weight $\widehat{\kappa}_{l_1,l_2}(\omega_1,\omega_2) = (\epsilon_{\omega_1,l_1},\, K\epsilon_{\omega_2,l_2})$ the corresponding integral operator K will enjoy certain bonus properties:

$$\widehat{\kappa}_{l_1,l_2}(\omega_1,\omega_2) = \frac{2\pi \widehat{\kappa}_{l_1,l_2}(\omega_1)}{\omega_1 \tanh \pi\omega_1} \delta(\omega_1 - \omega_2) \qquad \text{translation inv.} \tag{A.19a}$$

$$\widehat{\kappa}_{l_1,l_2}(\omega_1,\omega_2) = \delta_{l_1+l_2,0} \; \widehat{\kappa}_{l_1}(\omega_1,\omega_2) \qquad \mathrm{SO}^\uparrow(2) \;\; \text{inv.} \tag{A.19b}$$

$$\widehat{\kappa}_{l_1,l_2}(\omega_1,\omega_2) = \delta_{l_1+l_2,0} \; \frac{2\pi \widehat{\kappa}(\omega_1)}{\omega_1 \tanh \pi\omega_1} \delta(\omega_1 - \omega_2) \qquad \mathrm{SO}(1,2) \;\; \text{inv.} \tag{A.19c}$$

For K itself these properties amount to a vanishing commutator with $\mathbb{T}$, $\rho|_{\mathrm{SO}^\uparrow(2)}$ and ρ, respectively. The fact that the spectral weights (A.19c) lead to $\mathrm{SO}(1,2)$ invariant operators follows from (A.12c); the kernels $\kappa(n,n')$ of these operators depend on the invariant distance $n \cdot n'$ only.

As an example of a spectral decomposition consider the transfer matrix itself where the weights are just the eigenvalues (2.3). Using e.g. [28], p.804 and the completeness relation for the Legendre functions one verifies

$$\frac{\beta}{2\pi} \exp\{\beta(1 - n\cdot n')\} = \int_0^\infty \frac{d\omega}{2\pi} \, \omega \tanh \pi\omega \, \mathcal{P}_{-1/2+i\omega}(n\cdot n') \, \lambda_\beta(\omega) \,. \tag{A.20}$$

In representation theoretical terms this expresses the exponential of a singlet wrt the non-unitary vector irrep as a superposition of singlets wrt the unitary irrep (A.15). We note that the inverse Mehler-Fock transform (A.16) gives

$$\lambda_\beta(\omega) = \beta \int_1^\infty d\xi e^{\beta(1-\xi)} \mathcal{P}_{-1/2+i\omega}(\xi). \tag{A.21}$$

Clearly the integral kernels $\kappa(n \cdot n')$ that give rise to well-defined operators on $L^2(\mathbb{H})$ must have suitable regularity and decay properties. The asymptotics in (A.13) suggests

that the kernels $\kappa(\xi)$ should also decay at least like $\xi^{-1/2}$. Some decay stronger than $\xi^{-1/2}$ is also necessary in order for κ to be the integral kernel of a densely defined *operator* from $L^2(\mathbb{H})$ to $L^2(\mathbb{H})$. A sufficient condition seems to be more difficult to obtain, but in any case kernels like $n \cdot n'$ do *not* correspond to densely defined operators on $L^2(\mathbb{H})$ (they give rise only to densely defined quadratic forms).

The integrands of the Legendre functions (A.11) likewise provide eigenfunctions of the Laplace-Beltrami operator (A.6). Explicitly

$$E_{\omega,u}(n) := E_{\omega,u}(\xi,\varphi) = [\xi - \sqrt{\xi^2-1}\cos(u-\varphi)]^{-1/2-i\omega}\,, \tag{A.22}$$

are bounded complex solutions for all $|\varphi - u(\mathrm{mod}2\pi)| > \epsilon > 0)$, decaying like $\xi^{-1/2+i\omega}$ for $\xi \to \infty$. The upper bound will diverge as $\epsilon \to 0$ because for $\phi = u$ one has $|E_{\omega,u}(\xi,u)| \sim \sqrt{2\xi}$, for $\xi \to \infty$. The Legendre functions (A.11) are recovered as the Fourier modes of (A.22) and vice versa. The orthogonality and completeness relations take the form

$$\begin{aligned} &(E_{\omega,\theta},\, E_{\omega',\theta'}) = \frac{(2\pi)^2}{\omega\tanh\pi\omega}\delta(\omega-\omega')\delta(\theta-\theta')\,, \\ &\frac{1}{(2\pi)^2}\int_0^\infty d\omega\,\omega\tanh\omega\int_0^{2\pi} d\theta\, E_{\omega,\theta}(n)^* E_{\omega,\theta}(n') = \delta(n,n')\,. \end{aligned} \tag{A.23}$$

The main virtue of these solutions is their simple transformation law under SO$(1,2)$, see e.g. [45]. For a boost $A^{-1} = A(\theta,\alpha)^{-1}$ mapping $\xi = n^0$ into $\xi\mathrm{ch}\theta - \mathrm{sh}\theta\cos(\varphi-\alpha)$ one has

$$E_{\omega,u}(A^{-1}n) = [\mathrm{ch}\theta + \cos(\varphi-\alpha)\mathrm{sh}\theta]^{-1/2-i\omega}E_{\omega,u'}(n)\,, \tag{A.24}$$

for some angle $u' = u'(\theta,\alpha)$. This is also a convenient starting point to show that the Fourier decomposition (A.14) indeed has the representation theoretical significance (A.15), see e.g. [43].

Appendix B: Flat noncompact spin chain

In order to elucidate the relation between the original Hilbert space and the one obtained by Osterwalder-Schrader reconstruction it is useful to consider the simplest noncompact spin chain where the target space is $\mathbb{R}$. The symmetry group in this case is also $\mathbb{R}$, which in contrast to SO(1,2) is amenable. Some of the unusual aspects of this model have been analyzed already in [21]. Of course all results generalize trivially to target spaces $\mathbb{R}^n$, $n > 1$.

We consider the Hilbert space $L^2(\mathbb{R})$ and take as the one-step transfer matrix simply the heat kernel $\exp[\beta^{-1}\Delta](u,v)$, so that

$$
\begin{aligned}
(\mathbb{T}^x\psi)(u) &= \int_{-\infty}^{\infty} dv\, \mathcal{T}_\beta(u-v;x)\,\psi(v)\,, \quad x\in\mathbb{N}\,,\\
\mathcal{T}_\beta(u;x) &= \sqrt{\frac{\beta}{2\pi x}}\exp\left[-\frac{\beta}{2x}\,u^2\right]. \qquad \text{(B.1)}
\end{aligned}
$$

The transfer operator trivially commutes with the action of $\mathbb{R}$ on the wave functions, i.e. $\mathbb{T}\circ\rho=\rho\circ\mathbb{T}$, with $\rho(a)\psi(u)=\psi(u-a)$. It is well known that the spectrum of $\mathbb{T}$ is continuous and covers the interval $[0,1]$; the generalized eigenfunctions are imaginary exponentials. As in the hyperbolic case, a gauge fix is necessary; we simply fix the leftmost 'spin' u_{-L} to 0, which is analogous to fixing $n_{-L}=n^{\uparrow}$. For the purpose of the Osterwalder-Schrader reconstruction we choose again in addition the bc $u_L=0$, i.e. we choose 0 Dirichlet conditions. As observable algebra we take $\mathcal{C}=\mathcal{C}_b$, the algebra of continuous bounded functions of finitely many variables $u_{x_1},\ldots u_{x_\ell}$, and we introduce the subalgebras $\mathcal{C}_+\,,\mathcal{C}_-$ and $\mathcal{C}_0=\mathcal{C}_+\cap\mathcal{C}_-$ as in Section 5.

For a finite chain the reconstruction of the Hilbert space $\mathcal{H}_L$ proceeds as in Section 5; we define for each $\mathcal{O}\in\mathcal{C}_+$

$$
\mathcal{O}_{0,L}(u_0)=\int\prod_{i=1}^{\ell}du_i\,\mathcal{O}(u_1,\ldots,u_\ell)\prod_{i=1}^{\ell}\mathcal{T}_\beta(u_i-u_{i-1};x_i-x_{i-1})\frac{\mathcal{T}_\beta(u_\ell;L-x_\ell)}{\mathcal{T}_\beta(u_0;L)}\,, \qquad \text{(B.2)}
$$

and

$$
\psi^{\mathcal{O}}(u)=\mathcal{O}_{0,L}(u)\frac{\mathcal{T}_\beta(u;L)}{\sqrt{\mathcal{T}_\beta(0;2L)}}\,. \qquad \text{(B.3)}
$$

The reconstructed Hilbert space $\mathcal{H}_L$ is the completion of $\mathcal{C}_0$ wrt ω_L in (B.5). It can be identified with the original $L^2(\mathbb{R})$ by the isometry

$$
V_L:\mathcal{H}_L\;\longrightarrow\;L^2(\mathbb{R})\,,\qquad (V_L\psi)(n)=\psi(u)\,\frac{\mathcal{T}_\beta(u;L)}{\sqrt{\mathcal{T}_\beta(0;2L)}}\,. \qquad \text{(B.4)}
$$

Equivalently $\mathcal{H}_L$ can be viewed as the preimage of $L^2(\mathbb{R})$ wrt V_L.

The thermodynamic limit can be readily understood here. The ratio $\mathcal{T}_\beta(u;L)/\mathcal{T}(0;L)$ approaches a constant for $L\to\infty$, signaling a unique ground state. On elements $\mathcal{O}_{0,\infty}\in\mathcal{C}_0$ the expectation functionals becomes an invariant mean, which exists in this case. There is a subspace $\mathcal{H}_{AP}$ of almost periodic functions on which this mean is unique, see [4]; this subspace consists of the completion (in the Hilbert space norm defined by the mean) of the space of trigonometric polynomials. A brief account of the theory of almost periodic functions on $\mathbb{R}$, which is due to H. Bohr, can be found in [48]. For $\psi\in\mathcal{H}_{AP}$ the mean is

$$
\omega(\psi)=\lim_{L\to\infty}\frac{1}{\sqrt{2\pi L}}\int du\,\exp\Big(-\frac{u^2}{2L}\Big)\psi(u)=:\lim_{L\to\infty}\omega_L(\psi)\,. \qquad \text{(B.5)}
$$

A better known expression of the invariant mean on $\mathcal{H}_{AP}$ is

$$\omega(\psi) = \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} du\, \psi(u)\,, \tag{B.6}$$

see for instance [48]. By the uniqueness these two expressions have to be the same for an almost periodic ψ and it is straightforward to verify this equivalence for the dense subspace of trigonometric polynomials. The scalar product induced by this invariant mean can be written as

$$(\psi', \psi)_{OS} = \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} du\, \psi(u)^* \psi(u)\,, \tag{B.7}$$

and the unitarity of ρ on $\mathcal{H}_{AP}$ is manifest.

It might be surprising that the Hilbert space obtained by the OS reconstruction from $\mathcal{C}_0$ is nonseparable; but it is well known that already the space $\mathcal{H}_{AP}$ is nonseparable [48]: there is an uncountable set of mutually orthonormal functions, namely the set

$$\{\psi_\alpha(u) = e^{i\alpha u} \mid \alpha \in \mathbb{R}\}\,. \tag{B.8}$$

One can introduce a shift automorphism τ like the one used in the hyperbolic case. From this one obtains a reconstructed transfer operator $\mathbb{T}_{OS}$ acting on $\mathcal{H}_{OS}$; in this case it is nonnegative and has again norm 1. $\mathbb{T}_{OS}$ acts on $\mathcal{C}_0$ simply by Eq. (B.1). This shows that the functions (B.8) are eigenvectors (in the proper sense) of $\mathbb{T}_{OS}$ with eigenvalue $\exp(-\frac{1}{2\beta}\alpha^2)$.

The relation between the original system $(L^2(\mathbb{R}), \mathbb{T})$ and the reconstructed one $(\mathcal{H}_{OS}, \mathbb{T}_{OS})$ turns out to be simply that the spectrum as a set remains the same, namely the interval $[0, 1]$. However there is now pure point spectrum on every point of the spectral interval and the generalized eigenfunctions become normalizable eigenstates. With respect to the representation of symmetry group $\mathbb{R}$ the original $L^2(\mathbb{R})$ is a direct integral of the one-dimensional irreducible representations on the imaginary exponentials (B.8), whereas $\mathcal{H}_{AP}$ is (and hence $\mathcal{H}_{OS}$ contains) a direct sum over the continuous parameters α:

$$\mathcal{H}_{OS} \supset \mathcal{H}_{AP} = \bigoplus_{\alpha\in\mathbb{R}} \mathcal{H}_\alpha\,, \tag{B.9}$$

where $\mathcal{H}_\alpha$ is the one-dimensional Hilbert space spanned by $e^{i\alpha u}$.

Let us end this appendix with the remark that the space $\mathcal{H}_{AP}$, huge as it is, is still only a small subspace of the full space $\mathcal{H}_{OS}$. It turns out that there are uncountably many more functions orthogonal to the exponentials discussed so far, for instance the functions $p_\alpha(u) = |u|^{i\alpha}$. Using distributional Fourier transformation one can show that

$$(p_\alpha, \psi_{\alpha'})_{OS} = 0\,, \quad \forall \alpha \neq 0\,,\ \alpha' \in \mathbb{R}. \tag{B.10}$$

Presumably these functions belong to the continuous spectrum overlaying the point spectrum we have found.

Appendix C: Inner products on $\mathcal{H}^0_{OS}$ and $\mathcal{H}^{\rm p}_{OS}$

Here we derive the formulas (5.34) and (5.35) for the inner products on $\mathcal{H}^0_{OS}$ and $\mathcal{H}^{\rm p}_{OS}$. We begin with (5.34), i.e. $(\rho_\infty(A)\psi_0, \rho_\infty(B)\psi_0)_{OS} = \mathcal{P}_{-1/2}(An^\uparrow\!\cdot\! Bn^\uparrow)$. By (4.12) this is equivalent to

$$\begin{aligned}
\lim_{n^\uparrow\cdot n\to\infty} f_{A,B}(n^\uparrow\cdot n) &= \mathcal{P}_{-1/2}(An^\uparrow\!\cdot\! Bn^\uparrow)\,, \quad \text{with} \\
f_{A,B}(n^\uparrow\cdot n) &:= \overline{\left(\frac{\mathcal{P}_{-1/2}(n\!\cdot\! An^\uparrow)}{\mathcal{P}_{-1/2}(n\!\cdot\! n^\uparrow)}\right)\left(\frac{\mathcal{P}_{-1/2}(n\!\cdot\! Bn^\uparrow)}{\mathcal{P}_{-1/2}(n\!\cdot\! n^\uparrow)}\right)}\,,
\end{aligned} \tag{C.1}$$

where the bar as before denotes the average over $\mathrm{SO}^\uparrow(2)$. Writing $\xi = n\cdot n^\uparrow$ and momentarily $n\cdot An^\uparrow = \xi\xi_A - \sqrt{\xi^2-1}\sqrt{\xi_A^2-1}\cos(\varphi-\varphi_A)$, and similarly for $n\cdot Bn^\uparrow$, the $\mathrm{SO}^\uparrow(2)$ average evaluates by means of (A.12c) to

$$f_{A,B}(\xi) = \sum_{l\in\mathbb{Z}} e^{-il(\varphi_A-\varphi_B)}\mathcal{P}^l_{-1/2}(\xi_A)\mathcal{P}^{-l}_{-1/2}(\xi_B)\left(\frac{\mathcal{P}^l_{-1/2}(\xi)\mathcal{P}^{-l}_{-1/2}(\xi)}{\mathcal{P}_{-1/2}(\xi)^2}\right). \tag{C.2}$$

The series converges uniformly in ξ: using the Cauchy-Schwarz inequality, the geometric-arithmetic mean inequality and the bound $|\mathcal{P}^l_{-1/2}(\xi)\mathcal{P}^{-l}_{-1/2}(\xi)| \leq \mathcal{P}_{-1/2}(\xi)^2$ the rhs is bounded by 1 and likewise the tail of the sum can be bounded uniformly in ξ. Taking now the limit $\xi\to\infty$ under the sum, which is permitted because of the uniform convergence of the series, one obtains

$$\lim_{\xi\to\infty} f_{A,B}(\xi) = \sum_{l\in\mathbb{Z}} e^{il(\varphi_A-\varphi_B)}(-)^l\mathcal{P}^l_{-1/2}(\xi_A)\mathcal{P}^{-l}_{-1/2}(\xi_B) = \mathcal{P}_{-1/2}(An^\uparrow\!\cdot\! Bn^\uparrow)\,, \tag{C.3}$$

using (A.13) and (A.12c). This gives (5.34); note that the result coincides with the one obtained from the 'correlated' limit in (5.13).

The derivation of (5.35) we break up in several steps. Recall the notation $\psi_\alpha(n) = \exp(i\alpha n^\uparrow\cdot n)$, $\alpha\in\mathbb{R}\setminus\{0\}$. We first show that these functions form an orthonormal system

$$(\psi_\alpha,\psi_\alpha)_{OS} = 1\,, \qquad (\psi_\alpha,\psi_{\alpha'})_{OS} = 0\,, \quad \text{for } \alpha\neq\alpha'\,. \tag{C.4}$$

The normalization is clear. For the orthogonality consider for $\alpha\neq 0$

$$I_\alpha(L) := \int_1^\infty d\xi\, e^{i\alpha\xi}\,\frac{\mathcal{T}_\beta(\xi;L)^2}{\mathcal{T}_\beta(1;2L)}\,. \tag{C.5}$$

To analyze this expression we integrate by parts and obtain

$$I_\alpha(L) = -\frac{\mathcal{T}_\beta(1;L)^2}{i\alpha\mathcal{T}_\beta(1;2L)}\left\{e^{i\alpha} + \int_1^\infty d\xi e^{i\alpha\xi}\,\frac{\partial}{\partial\xi}\left(\frac{\mathcal{T}_\beta(\xi;L)}{\mathcal{T}_\beta(1;L)}\right)^2\right\}\,. \tag{C.6}$$

The first term is $O(L^{-3/2})$ by (2.36); the modulus of the second term can be bounded, using the monotonicity of $\mathcal{T}_\beta(\xi;L)$ by

$$-\frac{\mathcal{T}_\beta(1;L)^2}{\alpha\mathcal{T}_\beta(1;2L)}\int_1^\infty d\xi\frac{\partial}{\partial\xi}\left(\frac{\mathcal{T}_\beta(\xi;L)}{\mathcal{T}_\beta(1;L)}\right)^2 = \frac{\mathcal{T}_\beta(1;L)^2}{\alpha\mathcal{T}_\beta(1;2L)}\,, \tag{C.7}$$

which is also $O(L^{-3/2})$. Together, $\lim_{L\to\infty} I_\alpha(L) = 0$ and (C.4) is proven.

We remark that this construction readily generalizes to all wave functions oscillating 'sufficiently fast' as $\xi \to \infty$. Consider

$$\psi_p(n) = \exp\left\{i\int_1^\xi du\, p(u)\right\} \quad \text{with} \quad \lim_{\xi\to\infty}\frac{(\ln\xi)^2}{\xi p(\xi)} = 0\,. \tag{C.8}$$

Then every pair of wave functions $\psi_{p_1}(n)$, $\psi_{p_2}(n)$, where the difference $p_1(\xi) - p_2(\xi)$ is strictly monotonous for suffiently large ξ and obeys the decay condition in (C.8) is orthogonal: $(\psi_{p_1}, \psi_{p_2})_{OS} = 0$, using Lemma 2.2 (iii) to get bounds uniform in L for the $\xi \to \infty$ limits. For example $\exp\{i\alpha(\ln\xi)^4\}$, $\alpha \in \mathbb{R}$, provides another nondenumerable orthonormal family, each member of which is orthogonal to each of the plain exponentials in (C.4). Here we shall only pursue the plain exponentials ψ_α, $\alpha \in \mathbb{R}$, further.

Repeating the above computations with the transformed exponentials $\rho_\infty(A)\psi_\alpha$ one readily shows that they remain orthogonal if they were initially. For the computation of the norms the phases are irrelevant, so they remain unity if $(\rho_\infty(A)\psi_0, \rho_\infty(A)\psi_0)_{OS} = (\psi_0, \psi_0)_{OS} = 1$. This however is a special case of (5.34). Thus

$$(\rho_\infty(A)\psi_\alpha, \rho_\infty(A)\psi_{\alpha'})_{OS} = (\psi_\alpha, \psi_{\alpha'})_{OS}\,, \quad \alpha, \alpha' \in \mathbb{R}\,. \tag{C.9}$$

In a last step we show

$$(\rho_\infty(A)\psi_\alpha\,,\, \psi_{\alpha'})_{OS} = 0\,, \qquad \forall\, A \in \mathrm{SO}(1,2)\,,\;\; \alpha\,, \alpha' \in \mathbb{R}\,,\;\; \alpha\alpha' \neq 0\,. \tag{C.10}$$

By definition one has

$$\begin{aligned}(\rho_\infty(A)\psi_\alpha\,,\, \psi_{\alpha'})_{OS} \;&=\; \lim_{L\to\infty} 2\pi\int_1^\infty d\xi\, e^{i\alpha'\xi} J_\alpha(\xi,\xi_A)\,\frac{\mathcal{T}_\beta(\xi,L)^2}{\mathcal{T}_\beta(1,2L)}\,, \\ J_\alpha(\xi,\xi_A) \;&:=\; \int_0^{2\pi}\frac{d\varphi}{2\pi}\, e^{-i\alpha A n^\uparrow\cdot n}\,\frac{\mathcal{P}_{-1/2}(An^\uparrow\cdot n)}{\mathcal{P}_{-1/2}(\xi)}\,,\end{aligned} \tag{C.11}$$

where we view $An^\uparrow\!\cdot n = \xi\xi_A - (\xi^2-1)^{1/2}(\xi_A^2-1)^{1/2}\cos(\varphi-\varphi_A)$ as a function of ξ, ξ_A and $\varphi - \varphi_A$. As anticipated by the notation $J_\alpha(\xi,\xi_A)$ is independent of φ_A. Clearly $J_\alpha(\xi,1) = e^{-i\alpha\xi}$ and $|J_\alpha(\xi,\xi_A)| \le \mathcal{P}_{-1/2}(\xi_A)$ by the addition theorem (A.12c). We take

now $\xi_A > 1$ and by (C.9) we may also assume that $\alpha \neq 0$ and wlog $\alpha > 0$ (while $\alpha' \in \mathbb{R}$ may be zero). By the argument familiar from section 4.2 only the behavior of $J_\alpha(\xi, \xi_A)$ for large ξ will be relevant for the inner product (C.11). We claim that

$$\begin{aligned} J_\alpha(\xi, \xi_A) &\sim \frac{1}{\sqrt{\alpha\xi}}\Big(Q_+(\xi_A)e^{-i\alpha p_+(\xi_A)\xi} + Q_-(\xi_A)e^{-i\alpha p_-(\xi_A)\xi}\Big) \quad \text{as } \xi \to \infty\,, \\ \text{with} \quad p_\pm(\xi_A) &= \xi_A \pm \sqrt{\xi_A^2 - 1}\,, \end{aligned} \tag{C.12}$$

and some complex constants $Q_\pm(\xi_A)$ nowhere zero for $\xi_A > 1$. Note that $1 < \xi_A < p_+(\xi_A)$ and $0 < p_-(\xi_A) < 1$.

We first show that the rhs of (C.12) is the leading term in an asymptotic expansion of $J_\alpha(\xi, \xi_A)$ for large ξ. The point to observe is that from (A.13) we have

$$\mathcal{P}_{-1/2}\Big(\xi\xi_A - (\xi^2-1)^{1/2}(\xi_A^2-1)^{1/2}\cos\varphi\Big)\mathcal{P}_{-1/2}(\xi)^{-1} \sim \frac{1}{[\xi_A - \sqrt{\xi_A^2-1}\cos\varphi]^{1/2}}\,, \tag{C.13}$$

with additive corrections of $O(1/\ln\xi)$. Asymptotically the integral becomes

$$J_\alpha(\xi, \xi_A) \sim e^{-i\alpha\xi\xi_A}\int_0^{2\pi}\frac{d\varphi}{2\pi}\frac{e^{i\alpha\xi\sqrt{\xi_A^2-1}\cos\varphi}}{[\xi_A - \sqrt{\xi_A^2-1}\cos\varphi]^{1/2}}\,. \tag{C.14}$$

For large ξ this integral can now be evaluated by the method of stationary phase (see e.g. [29]) with the result (C.12). The constants $Q_\pm(\xi_A)$ come out as

$$Q_\pm(\xi_A) = 2^{\pm 1/2}e^{\pm i\pi/4}\Big[2\pi\sqrt{\xi_A^2-1}\Big(\xi_A \pm \sqrt{\xi_A^2-1}\Big)\Big]^{-1/2}\,. \tag{C.15}$$

Subleading terms in the asymptotic expansion of $J_\alpha(\xi, \xi_A)$ could be worked out similarly, but are not needed. The properties relevant in the following are that $|J_\alpha(\xi, \xi_A)|$ vanishes for $\xi \to \infty$, and that the phases are linear in ξ with the given frequencies. To make sure that these are properties of $J_\alpha(\xi, \xi_A)$ and not just of its asymptotic expansion, we verified them numerically.

With (C.12) at our disposal, the rest of the derivation of (C.10) is straightforward. Substituting (C.12) into (C.11) one shows for generic ξ_A the vanishing of the $L \to \infty$ limit along the lines of (C.5) – (C.7). If both α and α' are nonzero one of the ξ-dependent phases might cancel for the special boost parameter $\xi_A = \frac{1}{2}(\frac{\alpha}{\alpha'} + \frac{\alpha'}{\alpha})$. The modulus of this term in the asymptotics of $J_\alpha(\xi, \xi_A)$ then is proportional to $(\alpha\xi)^{-1/2}$ which is an element of $\mathcal{C}^{\uparrow}_{\rm ainv}$, and the $L \to \infty$ limit vanishes on account of (4.12). This establishes (C.10). The result (5.35) then follows by combining (C.4), (C.9), and (C.10).

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